5.23 Spherical coordinates and vector differentiation via definition. (Section 6.3).

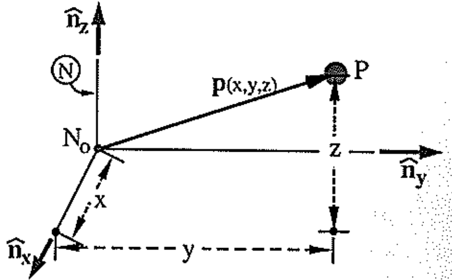
(a) Referring to Homework 5.22, use the definition of a vector derivative [equation (6.3)] to find the time-derivative in N of $\vec{p}$ and express it in terms of ρ, θ, ϕ, $\dot{\rho}$, $\dot{\theta}$, $\dot{\phi}$, and $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$.

Result:

$$\frac{{}^{N}d\vec{p}}{dt} = \left[\dot{\rho}\,\sin(\theta)\,\sin(\phi) + \rho\,\cos(\theta)\,\sin(\phi)\,\dot{\theta} + \rho\,\sin(\theta)\,\cos(\phi)\,\dot{\phi}\right]\,\hat{n}_x$$

$$+ \qquad\qquad \hat{n}_y$$

$$+ \qquad\qquad \hat{n}_z$$



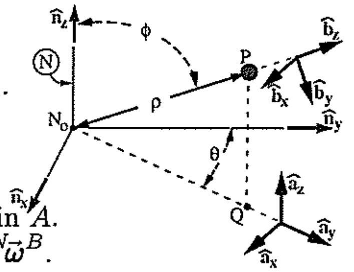
(b) Calculate $\frac{{}^{N}d\vec{p}}{dt}$ by using the ${}^{n}R^{b}$ rotation table to express $\frac{{}^{N}d\vec{p}}{dt}$ in terms of $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ and then doing <u>laborious</u> trigonometric simplifications. (Attempt this until it is clear how laborious this is.)

Result:

$$\frac{{}^{N}d\vec{p}}{dt} = \rho\,\sin(\phi)\,\dot{\theta}\,\hat{b}_x + \rho\dot{\phi}\,\hat{b}_y + \dot{\rho}\,\hat{b}_z$$

(c) The expression for $\frac{{}^{N}d\vec{p}}{dt}$ is simpler when expressed in terms of $(\hat{b}_x, \hat{b}_y, \hat{b}_z)$ / $(\hat{n}_x, \hat{n}_y, \hat{n}_z)$.

5.24 Spherical coordinates and vector differentiation via angular velocity.



(a) **Inspect** the figure to determine P's position vector from N_o.
Calculate $\vec{p}$'s time-derivative in B. Express results in terms of $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$.

Results: $\vec{p} = \quad \hat{b}_z \qquad \frac{{}^{B}d\vec{p}}{dt} =$

(b) Given below are A's angular velocity in N and B's angular velocity in A.
Complete the ***angular velocity addition theorem*** (below) to find ${}^{N}\vec{\omega}^{B}$.

Result:

$${}^{N}\vec{\omega}^{A} = -\dot{\theta}\,\hat{a}_z \qquad {}^{A}\vec{\omega}^{B} = -\dot{\phi}\,\hat{b}_x \qquad {}^{N}\vec{\omega}^{B} = {}^{N}\vec{\omega}^{A} + {}^{A}\vec{\omega}^{B} = \quad \hat{a}_z +$$

(c) Use the ***golden rule for vector differentiation*** (shown below-left) to calculate the time-derivative of $\vec{p}$ in N. Express results in terms of $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$.

Result:
Just Calculate!

$$\frac{{}^{N}d\vec{p}}{dt} = \frac{{}^{B}d\vec{p}}{dt} + {}^{N}\vec{\omega}^{B} \times \vec{p} \qquad \frac{{}^{N}d\vec{p}}{dt} = \boxed{\rho\,\sin(\phi)\,\dot{\theta}}\,\hat{b}_x + \quad \hat{b}_y + \quad \hat{b}_z$$

(d) Relative to the definition of vector differentiation in Homework 5.23b, the golden rule for vector differentiation is an **easier/harder** way to calculate $\frac{{}^{N}d\vec{p}}{dt}$.

Homework 6. Chapter 7.
Angular velocity and angular acceleration

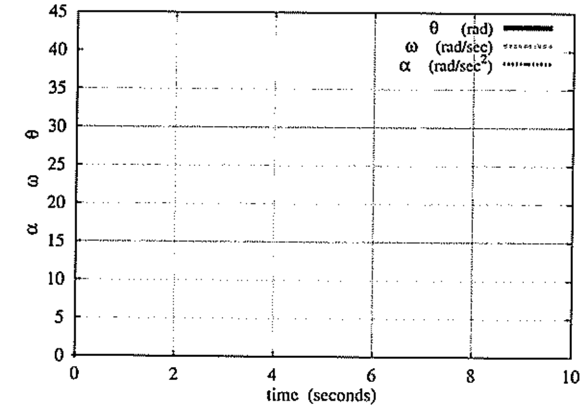
6.1 FE/EIT Review – Motion graph: $\boxed{T \Rightarrow \alpha \Rightarrow \omega \Rightarrow \theta}$

A wind turbine generates electricity from time-dependent aerodynamic wind forces. The wind creates a torque modeled as $T = 20\,\frac{\text{N m}}{\text{sec}} * t$.

Measures of the wind turbine's angular acceleration α, angular velocity ω, and angle θ are related by

$$\boxed{T \underset{(2D)}{=} I\,\alpha \qquad \alpha = \frac{d\omega}{dt} \qquad \omega \underset{(2D)}{=} \frac{d\theta}{dt}}$$

where $I = 80\,\text{kg}\,\text{m}^2$ is the relevant moment of inertia. Graph α in $\frac{\text{rad}}{\text{sec}^2}$, ω in $\frac{\text{rad}}{\text{sec}}$, and θ in rad for $0 \le t \le 10$ sec. Use initial values (i.e, values at $t = 0$) of $\omega = 0$ and $\theta = 0$.



6.2 Drawing a reference frame and unit vector bases. (Section 7.2)

<u>Draw</u> a reference frame or rigid body B, shaped like a uniform-density doughnut (having a hole).
<u>Draw</u> a right-handed orthogonal bases fixed in B having unit vectors $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$.
<u>Draw</u> a different right-handed orthogonal bases fixed in B with unit vectors $\hat{b}_1$, $\hat{b}_2$, $\hat{b}_3$.
<u>Draw</u> a properly located center of mass symbol ◐ and label this point as B_{cm}.
<u>Draw</u> a point B_o fixed on B, at a location different than B_{cm}.

6.3 Words and pictures for ${}^{b}R^{a}$, ${}^{N}\vec{\omega}^{B}$, ${}^{N}\vec{\alpha}^{B}$. (Chapters 5 and 7)

${}^{b}R^{a}$ – Description (words)	${}^{N}\vec{\omega}^{B}$ – Description (words)	${}^{N}\vec{\alpha}^{B}$ – Description (words)
Draw b and a	Draw B and N	

6.4 Definitions of angular velocity. (Section 7.3.3).

The definition of angular velocity of $\vec{\omega} \triangleq \dot{\theta}\,\vec{k}$ is a functional operational definition, i.e., in general, it is useful for calculating angular velocity and proving its properties (2D or 3D). **True/False**

6.5 Concept: What objects have a unique angular velocity/acceleration? (Sections 7.3, 7.4).

${}^N\vec{\omega}^S$, the angular velocity of an object S in a reference frame N is to be determined.

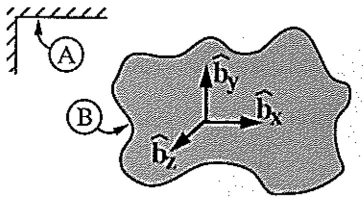
In general and **without ambiguity**, S could be a (circle *all* appropriate objects):

Real number	Point	Reference Frame	Mass center of a set of particles
Vector	Set of Points	Rigid Body	Mass center of a rigid body
Matrix	Particle	Flexible Body	Set of flexible bodies
Orthogonal unit basis	Set of Particles	Set of Rigid bodies	System of particles and bodies

Repeat for ${}^N\vec{\alpha}^S$, the angular acceleration of an object S in a reference frame N $\boxed{\text{box appropriate objects}}$.

6.6 Concepts: Angular velocity and unit vector directions. (Section 7.3 and Hw 6.11).

Right-handed orthogonal unit vectors $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ are fixed in a rigid body B and B's angular velocity in a reference frame A is (for all time)



$$ {}^A\vec{\omega}^B = 2\,\widehat{\mathbf{b}}_x + 3\,\widehat{\mathbf{b}}_y + 0\,\widehat{\mathbf{b}}_z $$

Statement: Since the $\widehat{\mathbf{b}}_z$ component of B's angular velocity in A is $\vec{0}$, $\widehat{\mathbf{b}}_z$'s direction does not change in A. **True/False**. (circle one).

Provide equation(s) that test whether or not the direction of $\widehat{\mathbf{b}}_z$ changes in A.

Equation:

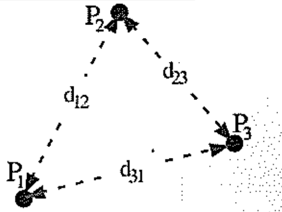
Hint: A mathematical test of whether the scalar variable y changes is $\frac{dy}{dt} \overset{?}{=} 0$.

6.7 What is a reference frame, rigid body, and orthogonal basis? (Sections 4.1 and 7.2)

#	Statement (regard "*rigid body*" as a massive 2D or 3D rigid object)	True or False
a	A reference frame has all the attributes of a rigid body.	**True/False**
b	A rigid body has all the attributes of a reference frame.	**True/False**
c	A reference frame with time-invariant distributed mass is a rigid body.	**True/False**
d	A massless rigid object is a reference frame.	**True/False**
e	The definition of a reference frame implies a sense of time.	**True/False**
f	A rigid body B may have an angular velocity in a reference frame N.	**True/False**
g	A point Q has a uniquely-defined angular velocity in a reference frame N.	**True/False**
h	The reference frame B implies unique orthogonal unit vectors $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$.	**True/False**
i	The right-handed orthogonal unit vectors $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ imply a unique reference frame.	**True/False**
j	The reference frame B implies a unique rigid frame.	**True/False**
k	A rigid frame with origin B_o and basis $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ implies a unique reference frame.	**True/False**

6.8 Concept: Reference frames and vector bases. (Sections 4.1 and 7.2)

Consider 3 distinct non-collinear points P_1, P_2, P_3 and the non-zero distances d_{12}, d_{23}, d_{31} between them. In general, determine if each object below can **always** be constructed from P_1, P_2, P_3 under the listed condition. For each "**Yes**" answer, draw the object.



Condition	Object to be constructed	Object can be constructed?	If Yes, **Draw**
d_{12}, d_{23}, d_{31} are **constant**	Vector basis that spans 3D space	Yes/No	
d_{12}, d_{23}, d_{31} are **variable**	Vector basis that spans 3D space	Yes/No	
d_{12}, d_{23}, d_{31} are **constant**	Right-handed, orthogonal, unitary basis	Yes/No	
d_{12}, d_{23}, d_{31} are **variable**	Right-handed, orthogonal, unitary basis	Yes/No	
d_{12}, d_{23}, d_{31} are **constant**	Unique reference frame	Yes/No	
d_{12}, d_{23}, d_{31} are **variable**	Unique reference frame	Yes/No	

6.9 Concepts: What objects have a uniquely-defined angular velocity? (Section 7.3).

#	For: ${}^A\vec{\omega}^B$ (B's angular velocity in A)	Object B		Object A	
a.	It is possible to find the angular velocity of a	point	in a	reference frame.	**True/False**
b.	It is possible to find the angular velocity of a	rigid body	in a	particle.	**True/False**
c.	It is possible to find the angular velocity of a	rigid body	in a	reference frame.	**True/False**
d.	It is possible to find the angular velocity of a	reference frame	in a	rigid body.	**True/False**
e.	It is possible to find the angular velocity of a	reference frame	in a	flexible body.	**True/False**
f.	It is possible to find the angular velocity of a	flexible body	in a	reference frame.	**True/False**

6.10 Rotational kinematics of a fire ladder. (Sections 7.3.3, 7.3.5, 7.3.6).

The following figure shows a fire truck chassis A traveling at constant speed in straight-line motion on Earth (A does not rotate relative to Earth). Earth is a *Newtonian reference frame* N.

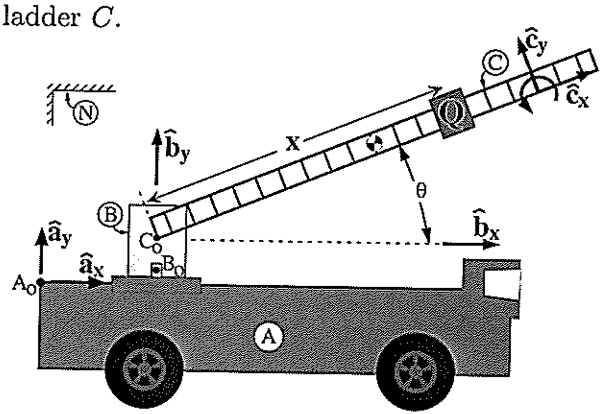
A rigid hub B is connected to fire truck A by a revolute motor at point B_o of B.

A rigid ladder C is connected to hub B by a revolute motor at point C_o of C .

A fire-fighter Q (modeled as a particle of mass m) climbs ladder C.

Right-handed orthogonal unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$; $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$; $\widehat{\mathbf{c}}_x$, $\widehat{\mathbf{c}}_y$, $\widehat{\mathbf{c}}_z$; are fixed in A, B, C, with:

- $\widehat{\mathbf{a}}_x$ pointing forward on the fire truck
- $\widehat{\mathbf{a}}_y$ vertically-upward and from B_o to C_o
- $\widehat{\mathbf{b}}_y = \widehat{\mathbf{a}}_y$ parallel to the axis of the revolute motor that connects B and A
- $\widehat{\mathbf{b}}_z = \widehat{\mathbf{c}}_z$ parallel to the axis of the revolute motor that connects B and C
- $\widehat{\mathbf{c}}_x$ directed from C_o to Q (along C's long axis)



Note: Visualize C's "**Body yz**" (or "**Space zy**") rotation sequence in N (e.g., with a ruler).

Quantity	Symbol	Type
$\widehat{\mathbf{b}}_y$ measure of B's angular velocity in A	ω_B	Constant
Angle from $\widehat{\mathbf{b}}_x$ to $\widehat{\mathbf{c}}_x$ with $+\widehat{\mathbf{c}}_z$ sense	θ	Variable

${}^{c}R^{b}$			

(a) Complete the previous ${}^{c}R^{b}$ rotation table (to the right).
Note: ${}^{c}R^{b}$ is unnecessary for the remainder of this problem.

(b) Clarify the process to determine ${}^B\vec{\omega}^C$, then express it in terms of $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$. (Section 7.3.3).

- C's angular velocity in B is simple since is fixed in both and .
- is the time-derivative of the angle between and .
- The sign (±) was determined using the -hand rule (sweep from to).
- ${}^B\vec{\omega}^C =$

(c) B's angular velocity in A is known to be a simple angular velocity of ${}^A\vec{\omega}^B = \omega_B\,\widehat{\mathbf{b}}_y$ because $\widehat{\mathbf{b}}_y$ is a vector fixed in both and .

(d) Form C's angular velocity in N and express it in terms of $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$.

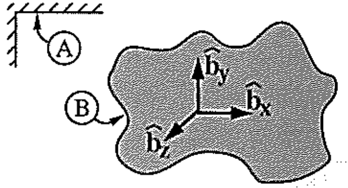
Result:
$$ {}^N\vec{\omega}^C \underset{(7.4)}{=} \vec{\omega} + \vec{\omega} + \vec{\omega} = \vec{0} + \quad \widehat{\mathbf{b}}_y + \quad \widehat{\mathbf{b}}_z $$

(e) When both ω_B and $\dot{\theta}$ are *constant*, ${}^N\vec{\alpha}^C = \vec{0}$. **True/False**.

(f) Write the definition for C's angular acceleration in N and form ${}^N\vec{\alpha}^C$. (Sections 7.4, 7.3).

Result:
$$ {}^N\vec{\alpha}^C \underset{(7.8)}{\triangleq} \qquad\qquad {}^N\vec{\alpha}^C \underset{(7.1)}{=} \omega_B\,\dot{\theta}\,\widehat{\mathbf{b}}_x + \ddot{\theta}\,\widehat{\mathbf{b}}_z $$

6.11 Concept: Angular velocity, angular acceleration, and a fixed vector. (Section 7.3 and Hw 6.6).



The figure to the right shows a rigid body B in a reference frame A. Orthogonal unit vectors $\widehat{b}_x$, $\widehat{b}_y$, $\widehat{b}_z$ are fixed in B. The following questions relate to B's angular velocity in A.

(a) Circle those expressions for ${}^{A}\vec{\omega}^{B}$ for which $\widehat{b}_z$ remains **constant** (fixed) in A. Complete the calculation below that helps verify your answer.

$$ {}^{A}\vec{\omega}^{B} = 3\,\widehat{b}_z \qquad {}^{A}\vec{\omega}^{B} = (3{+}t)\,\widehat{b}_z \qquad {}^{A}\vec{\omega}^{B} = 3\,\widehat{b}_x + 4\,\widehat{b}_y \qquad {}^{A}\vec{\omega}^{B} = 4\,\widehat{b}_y + t\,\widehat{b}_z $$

$$ \frac{{}^{A}d\widehat{b}_z}{dt} \underset{(7.1)}{=} \qquad + \qquad \times $$

(b) Circle the expressions for ${}^{A}\vec{\omega}^{B}$ that remain **constant** (fixed) in A. Complete the calculation below that helps verify your answer.

$$ {}^{A}\vec{\omega}^{B} = 3\,\widehat{b}_z \qquad {}^{A}\vec{\omega}^{B} = (3{+}t)\,\widehat{b}_z \qquad {}^{A}\vec{\omega}^{B} = 3\,\widehat{b}_x + 4\,\widehat{b}_y \qquad {}^{A}\vec{\omega}^{B} = 4\,\widehat{b}_y + t\,\widehat{b}_z $$

$$ \frac{{}^{A}d\,{}^{A}\vec{\omega}^{B}}{dt} \underset{(7.1)}{=} \qquad + \qquad \times $$

(c) Circle those expressions that result in B's angular acceleration in A being **non-zero**.

$$ {}^{A}\vec{\omega}^{B} = 3\,\widehat{b}_z \qquad {}^{A}\vec{\omega}^{B} = (3{+}t)\,\widehat{b}_z \qquad {}^{A}\vec{\omega}^{B} = 3\,\widehat{b}_x + 4\,\widehat{b}_y \qquad {}^{A}\vec{\omega}^{B} = 4\,\widehat{b}_y + t\,\widehat{b}_z $$

6.12 Theorems: Rotation matrices R, angular velocity $\vec{\omega}$, angular acceleration $\vec{\alpha}$? (Section 7.4).

Determine whether or not each theorem to the right is valid for general 3D motion of reference frames A, B, C, and D.

Theorem	True or false
${}^{a}R^{d} = {}^{a}R^{b} * {}^{b}R^{c} * {}^{c}R^{d}$	**True/False**
${}^{A}\vec{\omega}^{D} = {}^{A}\vec{\omega}^{B} + {}^{B}\vec{\omega}^{C} + {}^{C}\vec{\omega}^{D}$	**True/False**
${}^{A}\vec{\alpha}^{D} = {}^{A}\vec{\alpha}^{B} + {}^{B}\vec{\alpha}^{C} + {}^{C}\vec{\alpha}^{D}$	**True/False**

6.13 Alternate formula for angular acceleration. (Section 7.3).

Prove ${}^{N}\vec{\alpha}^{B} \triangleq \dfrac{{}^{N}d\;{}^{N}\vec{\omega}^{B}}{dt}$ can also be calculated as ${}^{N}\vec{\alpha}^{B} = \dfrac{{}^{B}d\;{}^{N}\vec{\omega}^{B}}{dt}$.

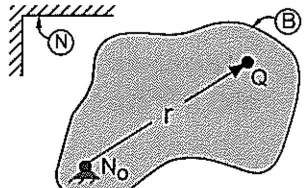
6.14 Concepts: Angular acceleration for general 3D motion. (Sections 7.3, 7.4).

Determine whether or not each of the following equations generally apply to the angular acceleration $\vec{\alpha}$ of reference frames A, B, and C in general 3D motion.

${}^{A}\vec{\alpha}^{B} = \dfrac{{}^{A}d\;{}^{A}\vec{\omega}^{B}}{dt}$	True/False	${}^{A}\vec{\alpha}^{C} = {}^{A}\vec{\alpha}^{B} + {}^{B}\vec{\alpha}^{C} + {}^{A}\vec{\omega}^{B} \times {}^{B}\vec{\omega}^{C}$	True/False
${}^{A}\vec{\alpha}^{B} = \dfrac{{}^{A}d\;{}^{B}\vec{\omega}^{A}}{dt}$	True/False	${}^{A}\vec{\alpha}^{B} = -\dfrac{{}^{A}d\;{}^{B}\vec{\omega}^{A}}{dt}$	True/False
${}^{A}\vec{\alpha}^{B} = \dfrac{{}^{C}d\;{}^{A}\vec{\omega}^{B}}{dt}$	True/False	${}^{A}\vec{\alpha}^{B} = \dfrac{{}^{C}d\;{}^{A}\vec{\omega}^{B}}{dt} + {}^{A}\vec{\omega}^{C} \times {}^{A}\vec{\omega}^{B}$	True/False
${}^{A}\vec{\alpha}^{B} = \dfrac{{}^{B}d\;{}^{A}\vec{\omega}^{B}}{dt}$	True/False	${}^{A}\vec{\alpha}^{B} = \dfrac{{}^{C}d\;{}^{A}\vec{\omega}^{B}}{dt} + {}^{B}\vec{\omega}^{C} \times {}^{A}\vec{\omega}^{B}$	True/False
${}^{A}\vec{\alpha}^{B} = {}^{B}\vec{\alpha}^{A}$	True/False	${}^{A}\vec{\alpha}^{B} = -{}^{B}\vec{\alpha}^{A}$	True/False

---

6.15 Vector differentiation concepts "$v = \omega\, r$". (Section 7.3).

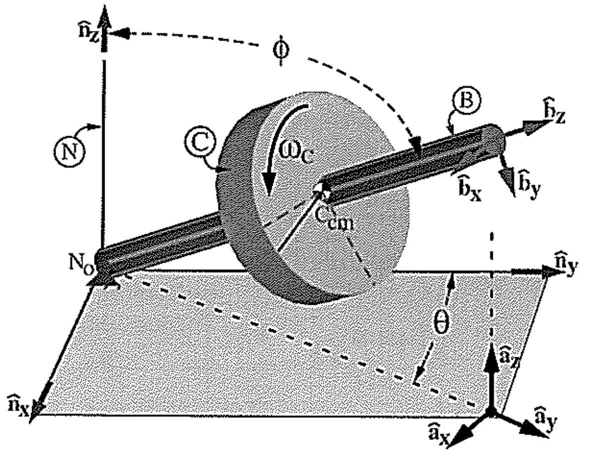


Point Q is fixed on a rigid body B. Point N_o is fixed in a reference frame N and does not move on B. Complete the following proof that shows how $\vec{v}$ (Q's velocity in N) can be written in terms of ${}^{N}\vec{\omega}^{B}$ (B's angular velocity in N) and $\vec{r}$ (Q's position vector from N_o).

Mathematical statement	Reasoning (explain each step in the proof with a brief phrase)
$\vec{v} \triangleq \dfrac{{}^{N}d\vec{r}}{dt}$	Definition of Q's velocity in N
$= \qquad +$	
$= {}^{N}\vec{\omega}^{B} \times \vec{r}$	

6.16 Angular velocity/acceleration of precessing, nutating, spinning, gyro. (Sections 7.3.3, 7.3.5, 7.4).

The following figure shows a gyro moving in a reference frame N. The gyro's cylindrical rotor C is supported in bearings by a gimbal B, so that C has a simple angular velocity in B of ${}^{B}\vec{\omega}^{C} = \omega_C\,\widehat{b}_z$. Gimbal B is set in N so one point of B is always coincident with point N_o fixed in N.



Right-handed sets of orthogonal unit vectors $\widehat{n}_x$, $\widehat{n}_y$, $\widehat{n}_z$; $\widehat{a}_x$, $\widehat{a}_y$, $\widehat{a}_z$; and $\widehat{b}_x$, $\widehat{b}_y$, $\widehat{b}_z$, are fixed in reference frames N, A, and B, respectively, with $\widehat{n}_z$ vertically-upward and $\widehat{b}_z$ directed along the rotor's axis and pointing from N_o to C_{cm} (C's center of mass).

The orientation of A in N is determined by initially setting $\widehat{a}_i = \widehat{n}_i$ $(i = x, y, z)$ and then subjecting A to a right-handed rotation characterized by $-\theta\,\widehat{a}_z$. Hence

$$ {}^{N}\vec{\omega}^{A} \underset{(7.2)}{=} -\dot{\theta}\,\widehat{a}_z $$

Gimbal B's orientation in A is found by initially setting $\widehat{b}_i = \widehat{a}_i$ $(i = x, y, z)$ and then subjecting B to a right-handed rotation characterized by $-\phi\,\widehat{b}_x$.

Note: Simplify this problem by creating the ${}^{b}R^{a}$ rotation table.

(a) **<u>Visualize</u>** C's orientation in N (<u>rotate</u> a pen C in proper <u>sequence</u>: **<u>first</u>** $-\theta\,\widehat{n}_z$, **<u>then</u>** $-\phi\,\widehat{a}_x$, then $\omega_C\,\widehat{b}_z$).[1]

(b) Form B and C's angular velocity in N (in terms of θ, ϕ, $\dot{\theta}$, $\dot{\phi}$, ω_C, and $\widehat{b}_x$, $\widehat{b}_y$, $\widehat{b}_z$).

Result:

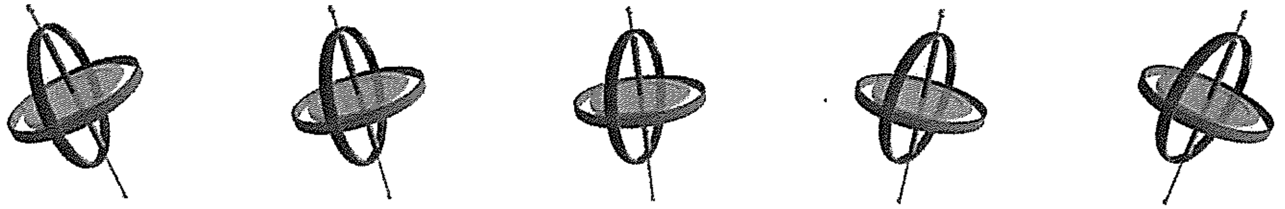
$$ {}^{N}\vec{\omega}^{B} \underset{(7.4)}{=} \vec{\omega} + \vec{\omega} = \qquad \widehat{b}_x + \qquad \widehat{b}_y + -\cos(\phi)\,\dot{\theta}\,\widehat{b}_z $$

$$ {}^{N}\vec{\omega}^{C} \underset{(7.4)}{=} \vec{\omega} + \vec{\omega} = \qquad \widehat{b}_x + \sin(\phi)\,\dot{\theta}\,\widehat{b}_y + \qquad \widehat{b}_z $$

(c) Express B's angular acceleration in N in terms of θ, $\dot{\theta}$, $\ddot{\theta}$, ϕ, $\dot{\phi}$, $\ddot{\phi}$, ω_C, $\dot{\omega}_C$, and $\widehat{b}_x$, $\widehat{b}_y$, $\widehat{b}_z$.

Result:

$$ {}^{N}\vec{\alpha}^{B} = -\ddot{\phi}\,\widehat{b}_x + \Big[\cos(\phi)\,\dot{\phi}\,\dot{\theta} + \sin(\phi)\,\ddot{\theta}\Big]\,\widehat{b}_y + \Big[\sin(\phi)\,\dot{\phi}\,\dot{\theta} - \cos(\phi)\,\ddot{\theta}\Big]\,\widehat{b}_z $$



---

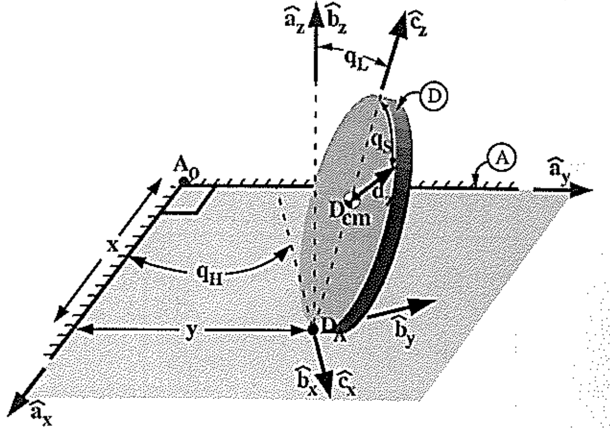
[1] This "**Body zxz**" rotation sequence of C in N is equivalent to a "**Space zxz**" rotation sequence.

6.17 Rotating disk angular velocity/acceleration (a first step for wheeled vehicles). (Sections 7.3.3, 7.3.5).

The figure to the right shows a thin disk D rotating on a horizontal plane A (Newtonian reference frame).

Right-handed orthogonal unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ and $\hat{\mathbf{d}}_x$, $\hat{\mathbf{d}}_y$, $\hat{\mathbf{d}}_z$ are fixed in A and D respectively, with $\hat{\mathbf{a}}_y$ horizontally-right, $\hat{\mathbf{a}}_z$ vertically-upward, and $\hat{\mathbf{d}}_y$ parallel to the disk's axis.

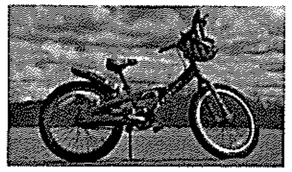
D's orientation in A is determined by initially setting $\hat{\mathbf{d}}_i = \hat{\mathbf{a}}_i$ $(i = x, y, z)$ and then subjecting D to a <u>sequence</u> of "body-fixed" right-handed rotations in A characterized by $q_H\,\hat{\mathbf{d}}_z$, $-q_L\,\hat{\mathbf{d}}_x$, $q_S\,\hat{\mathbf{d}}_y$.



This sequence of three rotations can be separated into three simple rotations as follows.

1. Initially orient right-handed orthogonal unit vectors $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ so $\hat{\mathbf{b}}_i = \hat{\mathbf{a}}_i$ $(i = x, y, z)$ and then subject B to a right-handed rotation in A characterized by $q_H\,\hat{\mathbf{a}}_z$.
2. Initially orient right-handed orthogonal unit vectors $\hat{\mathbf{c}}_x$, $\hat{\mathbf{c}}_y$, $\hat{\mathbf{c}}_z$ so $\hat{\mathbf{c}}_i = \hat{\mathbf{b}}_i$ $(i = x, y, z)$ and then subject C to a right-handed rotation in B characterized by $-q_L\,\hat{\mathbf{b}}_x$.
3. Initially orient $\hat{\mathbf{d}}_i = \hat{\mathbf{c}}_i$ $(i = x, y, z)$ and then subject D to a right-handed rotation of $q_S\,\hat{\mathbf{c}}_y$.

Name	Description	Symbol	Type
Heading angle	Angle from $\hat{\mathbf{a}}_x$ to $\hat{\mathbf{b}}_x$ with $+\hat{\mathbf{a}}_z$ sense	q_H	Variable
Lean angle	Angle from $\hat{\mathbf{b}}_z$ to $\hat{\mathbf{c}}_z$ with $-\hat{\mathbf{b}}_x$ sense	q_L	Variable
Spin angle	Angle from $\hat{\mathbf{c}}_z$ to $\hat{\mathbf{d}}_z$ with $+\hat{\mathbf{c}}_y$ sense	q_S	Variable



(a) <u>Visualize</u> D's orientation in A, e.g., <u>rotate</u> a DVD by the <u>sequence</u> of angles q_H then q_L then q_S. Sketch the missing vectors $\hat{\mathbf{c}}_y$, $\hat{\mathbf{d}}_x$, $\hat{\mathbf{d}}_y$ on the figure and form the $^{b}R^{a}$ and $^{c}R^{b}$ rotation tables.
Result: $^{b}R^{a}$ $^{c}R^{b}$

(b) Find D's angular velocity in A and then express it in terms of $\hat{\mathbf{c}}_x$, $\hat{\mathbf{c}}_y$, $\hat{\mathbf{c}}_z$.[2]
Result: $^{A}\vec{\omega}^{D} = \dot{q}_H\,\hat{\mathbf{b}}_z - \dot{q}_L\,\hat{\mathbf{c}}_x + \dot{q}_S\,\hat{\mathbf{c}}_y = -\dot{q}_L\,\hat{\mathbf{c}}_x + [\dot{q}_S - \sin(q_L)\,\dot{q}_H]\,\hat{\mathbf{c}}_y + \cos(q_L)\,\dot{q}_H\,\hat{\mathbf{c}}_z$

(c) For efficient kinematics, $^{A}\vec{\omega}^{D}$ is rewritten $^{A}\vec{\omega}^{D} = \omega_x\,\hat{\mathbf{c}}_x + \omega_y\,\hat{\mathbf{c}}_y + \omega_z\,\hat{\mathbf{c}}_z$ where ω_x, ω_y, ω_z are variables. Form ***kinematical differential equations*** relating $\dot{q}_L$, $\dot{q}_S$, $\dot{q}_H$ to ω_x, ω_y, ω_z.
Result: $\dot{q}_L = -\omega_x$ $\dot{q}_S = \omega_y + \tan(q_L)\,\omega_z$ $\dot{q}_H = \frac{\omega_z}{\cos(q_L)}$

(d) What value of q_L produces an indeterminate value for $\dot{q}_H$, $\dot{q}_L$, or $\dot{q}_S$?
Does this indeterminate value have any physical significance for this problem?
Result: $q_L = \;\;^\circ$ This corresponds to the disk laying flat on the plane.

(e) Using $^{A}\vec{\omega}^{D} = \omega_x\,\hat{\mathbf{c}}_x + \omega_y\,\hat{\mathbf{c}}_y + \omega_z\,\hat{\mathbf{c}}_z$, show $^{A}\vec{\alpha}^{D}$ and $^{A}\vec{\omega}^{C}$ are

D's angular acceleration in A $\quad ^{A}\vec{\alpha}^{D} = (\dot{\omega}_x - \omega_z\,\dot{q}_S)\,\hat{\mathbf{c}}_x + \dot{\omega}_y\,\hat{\mathbf{c}}_y + (\omega_x\,\dot{q}_S + \dot{\omega}_z)\,\hat{\mathbf{c}}_z$

C's angular velocity in A $\quad ^{A}\vec{\omega}^{C} = \omega_x\,\hat{\mathbf{c}}_x + (\omega_y - \dot{q}_S)\,\hat{\mathbf{c}}_y + \omega_z\,\hat{\mathbf{c}}_z = \omega_x\,\hat{\mathbf{c}}_x - \tan(q_L)\,\omega_z\,\hat{\mathbf{c}}_y + \omega_z\,\hat{\mathbf{c}}_z$

---

[2] The MotionGenesis command `Express( D.GetAngularVelocity(A), C )` expresses $^{A}\vec{\omega}^{D}$ in terms of $\hat{\mathbf{c}}_x$, $\hat{\mathbf{c}}_y$, $\hat{\mathbf{c}}_z$.

6.18 Optional: Rotating disk angular acceleration in terms of $\ddot{q}_H$, $\ddot{q}_L$, $\ddot{q}_S$. (Sections 7.3, 7.4).

Using $^{A}\vec{\omega}^{D}$ from Homework 6.17b, calculate D's angular acceleration in A.
Result:

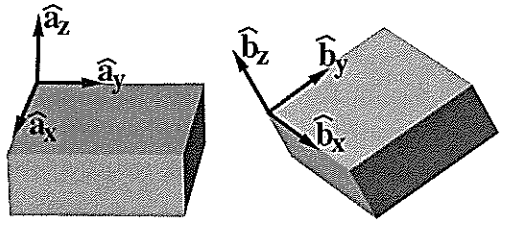
$$^{A}\vec{\alpha}^{D} = [-\cos(q_L)\,\dot{q}_H\,\dot{q}_S - \ddot{q}_L]\,\hat{\mathbf{c}}_x + [\ddot{q}_S - \cos(q_L)\,\dot{q}_H\,\dot{q}_L - \sin(q_L)\,\ddot{q}_H]\,\hat{\mathbf{c}}_y + [\cos(q_L)\,\ddot{q}_H - \dot{q}_L\,\dot{q}_S - \sin(q_L)\,\dot{q}_H\,\dot{q}_L]\,\hat{\mathbf{c}}_z$$

$^{A}\vec{\omega}^{D}$ is shorter when expressed in terms of $(\dot{q}_H, \dot{q}_L, \dot{q}_S)$ / $(\omega_x, \omega_y, \omega_z)$.
$^{A}\vec{\alpha}^{D}$ is shorter when expressed in terms of $(\ddot{q}_H, \ddot{q}_L, \ddot{q}_S)$ / $(\dot{\omega}_x, \dot{\omega}_y, \dot{\omega}_z)$.

Note: ***Lagrange's equations of motion*** (Chapter 26) are built on generalized coordinates (e.g. q_H, q_L, q_S) and are usually less efficient than ***Kane's equations of motion*** (Chapter 25) which are built on generalized speeds (e.g., ω_x, ω_y, ω_z).

6.19 Concept: Vectors, bases, and reference frames.

The figure to the right shows right-handed orthogonal unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ and $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ fixed in rigid objects A and B, respectively. The $^{a}R^{b}$ rotation matrix and B's angular velocity in A are shown below where θ_1, θ_2, θ_3 and ω_x, ω_y, ω_z are time-dependent variables.



$^{b}R^{a}$	$\hat{\mathbf{a}}_x$	$\hat{\mathbf{a}}_y$	$\hat{\mathbf{a}}_z$
$\hat{\mathbf{b}}_x$	$\cos(\theta_2)\cos(\theta_3)$	$\sin(\theta_3)\cos(\theta_1) + \sin(\theta_1)\sin(\theta_2)\cos(\theta_3)$	$\sin(\theta_1)\sin(\theta_3) - \sin(\theta_2)\cos(\theta_1)\cos(\theta_3)$
$\hat{\mathbf{b}}_y$	$-\sin(\theta_3)\cos(\theta_2)$	$\cos(\theta_1)\cos(\theta_3) - \sin(\theta_1)\sin(\theta_2)\sin(\theta_3)$	$\sin(\theta_1)\cos(\theta_3) + \sin(\theta_2)\sin(\theta_3)\cos(\theta_1)$
$\hat{\mathbf{b}}_z$	$\sin(\theta_2)$	$-\sin(\theta_1)\cos(\theta_2)$	$\cos(\theta_1)\cos(\theta_2)$

$$^{A}\vec{\omega}^{B} = \omega_x\,\hat{\mathbf{b}}_x + \omega_y\,\hat{\mathbf{b}}_y + \omega_z\,\hat{\mathbf{b}}_z$$

Calculate the time-derivative in reference frame A of the vector $\vec{s}$ (given below).
Express your results in terms of whatever symbols and unit vectors simplify your work.
Result: (Note: There is a <u>long</u>, <u>medium</u>, and <u>short</u> way to do this problem.)

$$\vec{s} = t\,\hat{\mathbf{a}}_x + t^2\,\hat{\mathbf{a}}_y + \hat{\mathbf{b}}_x + 2\,\hat{\mathbf{b}}_y$$

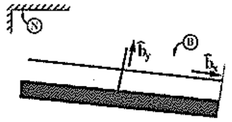
$$\frac{^{A}d\,\vec{s}}{dt} =$$

6.20 3D spin stability (application to Top-gun and Explorer I). (Sections 7.3 and 7.3.2)

The ***angular momentum principle*** for a rigid body B in a Newtonian reference frame N is

$$\boxed{\vec{M} \underset{(22.4)}{=} \frac{^{N}d\,\vec{H}}{dt}}$$

$\vec{M}$ is the moment of all forces on B about B_{cm} (B's center of mass)
$\vec{H}$ is B's angular momentum about B_{cm} in N



Right-handed unit vectors $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ are fixed in B and parallel to B's principal inertia axes about B_{cm}. $\vec{M}$ and B's angular velocity and angular momentum about B_{cm} in N are given as

$\vec{M} = \vec{0}$ (ignores air-resistance etc).

$^{N}\vec{\omega}^{B} = \omega_x\,\hat{\mathbf{b}}_x + \omega_y\,\hat{\mathbf{b}}_y + \omega_z\,\hat{\mathbf{b}}_z$

$\vec{H} = I_{xx}\,\omega_x\,\hat{\mathbf{b}}_x + I_{yy}\,\omega_y\,\hat{\mathbf{b}}_y + I_{zz}\,\omega_z\,\hat{\mathbf{b}}_z$

Quantity	Symbol	Value
B's moment of inertia for $\hat{\mathbf{b}}_x$	I_{xx}	1 kg m²
B's moment of inertia for $\hat{\mathbf{b}}_y$	I_{yy}	2 kg m²
B's moment of inertia for $\hat{\mathbf{b}}_z$	I_{zz}	3 kg m²
$\hat{\mathbf{b}}_x$ measure of $^{N}\vec{\omega}^{B}$	ω_x	Variable
$\hat{\mathbf{b}}_y$ measure of $^{N}\vec{\omega}^{B}$	ω_y	Variable
$\hat{\mathbf{b}}_z$ measure of $^{N}\vec{\omega}^{B}$	ω_z	Variable

Optional: Inertia is explained in Chapters 14 and 16

(a) Starting with the angular momentum principle, show how to differentiate $\vec{H}$ and form scalar equations involving ω_x, ω_y, ω_z. Next, solve the scalar equations for $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$.

Result:

$$0 = \mathrm{I}_{xx}\,\dot{\omega}_x + (\mathrm{I}_{zz} - \mathrm{I}_{yy})\,\omega_z\,\omega_y \qquad\qquad \dot{\omega}_x = $$

$$0 = \mathrm{I}_{yy}\,\dot{\omega}_y + (\mathrm{I}_{xx} - \mathrm{I}_{zz})\,\omega_x\,\omega_z \qquad \Rightarrow \qquad \dot{\omega}_y = $$

$$0 = \qquad\qquad\qquad\qquad \dot{\omega}_z = \left[(\mathrm{I}_{xx} - \mathrm{I}_{yy})\,\omega_y\,\omega_x\right] / \,\mathrm{I}_{zz}$$

(b) Using MotionGenesis (or MATLAB® or ...), solve these ODEs for $0 \le t \le 4$ with the initial values corresponding to plot **#2**. Plot t, ω_x, ω_y, ω_z (compare to plot "Spin about **intermediate** axis").[3]

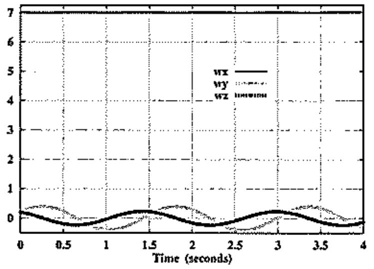
(c) The following plots show spin-stability about various axes.

Plot #	Initial values			Description	Stability
Plot 1	$\omega_x = 7$	$\omega_y = 0.2$	$\omega_z = 0.2$	Spin about minimum inertia axis	Neutral
Plot 2	$\omega_x = 0.2$	$\omega_y = 7$	$\omega_z = 0.2$	Spin about intermediate inertia axis	
Plot 3	$\omega_x = 0.2$	$\omega_y = 0.2$	$\omega_z = 7$	Spin about maximum inertia axis	

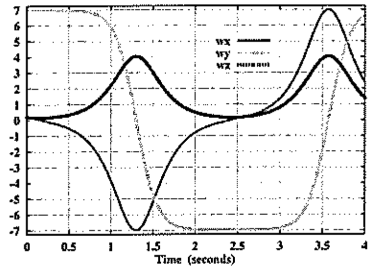
Using the following spinning book pictures, ***experimentally spin*** a book about each of its three axes (wrap rubber bands about the book so it stays closed) and use the experiment to complete the last column in the previous table with the word **unstable** or **neutral** or **stable**. with:

- **Unstable:** ω_x, ω_y, ω_z have large changes.
- **Neutral:** ω_x, ω_y, ω_z have small changes that do not increase or decrease much.
- **Stable:** ω_x, ω_y, ω_z have small changes that decrease to zero.

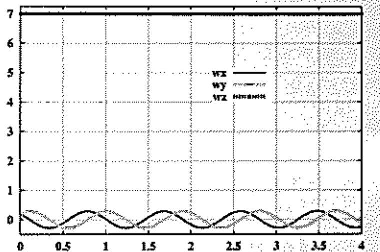
Spin about $\widehat{\mathrm{b}}_\mathrm{x}$ (**minimum** axis)

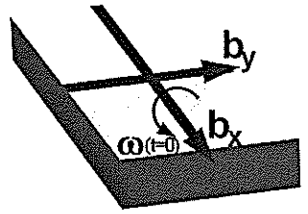


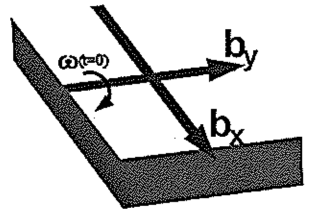
Spin about $\widehat{\mathrm{b}}_\mathrm{y}$ (**intermediate** axis)

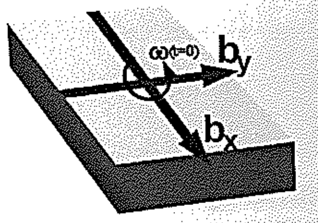


Spin about $\widehat{\mathrm{b}}_\mathrm{z}$ (**maximum** axis)

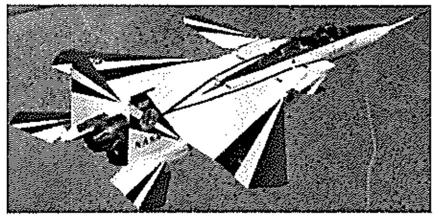


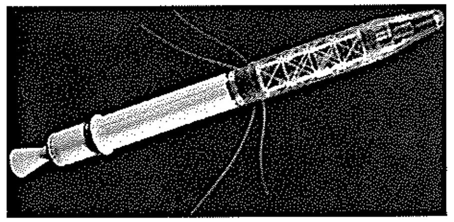


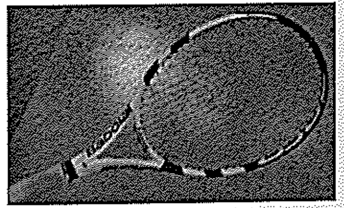




Note: This problem helps explain why "flat spin" is dangerous for aircraft (it is difficult to pull-out of spin about the maximum axis - featured in the Tom Cruise movie "Top-Gun"), why Explorer I (the first U.S. satellite) tumbled unstably on its first orbit, and why tennis racquets spin as they do.







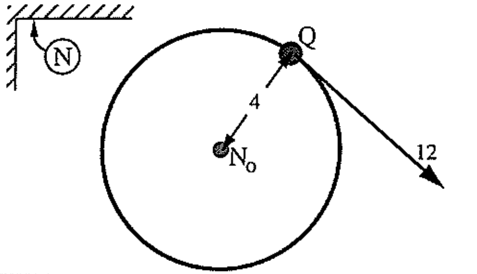
---

[3] Numerical solution of ODEs at: **www.MotionGenesis.com** ⇒ **Get Started** ⇒ **Solve Coupled 1st-order ODEs**. Analytical solution to these coupled nonlinear ODEs: Pgs. 187-195 of *Spacecraft Dynamics*, by Kane, Likins, and Levinson, McGraw-Hill, New York, 1985. Stability analysis of ODEs: *Control, Vibrations and Design of Dynamic Systems* by Mitiguy.

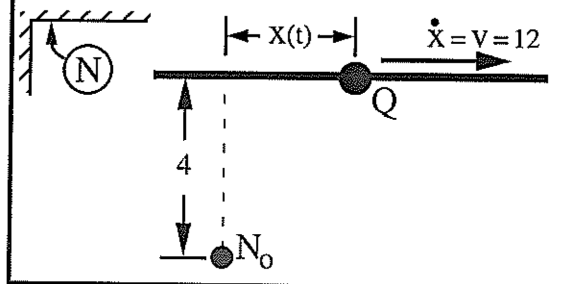
6.21 † Angular velocity concepts (2D motion)

The following figures show a point Q moving in a **plane** N. Point N_o is fixed in N. The left-figure shows Q moving clockwise with speed 12 on a circle of radius 4 (the circle is fixed in N and centered at N_o). The right-figure shows Q moving with a speed of 12 on a horizontal line that is a distance 4 from N_o. **Box** the following true statements about a uniquely-defined ***angular velocity*** for Q.

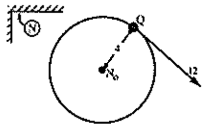
Q's angular velocity in N is $\vec{0}$.
Q's angular velocity in N is a non-zero vector.
Q's angular velocity in N is $\vec{\infty}$.
Q's angular velocity in N does not exist.



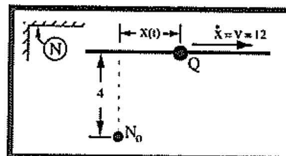
Q's angular velocity in N is $\vec{0}$.
Q's angular velocity in N is a non-zero vector.
Q's angular velocity in N is $\vec{\infty}$.
Q's angular velocity in N does not exist.



1. One can create a right-handed orthogonal unitary basis A consisting of $\widehat{\mathrm{a}}_\mathrm{x}$, $\widehat{\mathrm{a}}_\mathrm{y}$, $\widehat{\mathrm{a}}_\mathrm{z}$ with $\widehat{\mathrm{a}}_\mathrm{y}$ always directed from N_o to Q and $\widehat{\mathrm{a}}_\mathrm{z}$ outward normal to plane N. Calculate a **numerical** value for ${}^N\vec{\omega}^A$ for each situation below (Q on circle and Q on horizontal line).



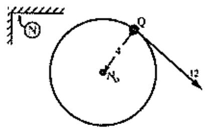
Numerical solution:
${}^N\vec{\omega}^A =$



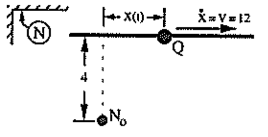
Numerical solution when $x = 3$:
${}^N\vec{\omega}^A = -1.92\,\widehat{\mathrm{a}}_\mathrm{z}$

For both situations, does ${}^N\vec{\omega}^A$ always exist, and if so, is it continuous when Q gradually decreases speed and reverses direction **Yes/No**. **Explain:**

2. A 2^{nd} possibility is to define a right-handed orthogonal vector basis B consisting of $\widehat{\mathrm{b}}_\mathrm{x}$, $\widehat{\mathrm{b}}_\mathrm{y}$, $\widehat{\mathrm{b}}_\mathrm{z}$ with $\widehat{\mathrm{b}}_\mathrm{x}$ always in the direction of Q's velocity in N and $\widehat{\mathrm{b}}_\mathrm{z}$ outward normal to N. Calculate a **numerical** value for ${}^N\vec{\omega}^B$ for each situation (Q on circle and Q on horizontal line).



Numerical solution:
${}^N\vec{\omega}^B =$



Numerical solution:
${}^N\vec{\omega}^B =$

For both situations, does ${}^N\vec{\omega}^B$ always exist, and if so, is it continuous when Q gradually decreases speed and reverses direction **Yes/No**. **Explain:**

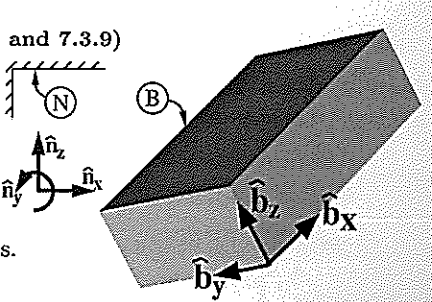
3. A 3^{rd} possibility is $\vec{\omega} = \dfrac{\vec{\mathrm{r}} \times \vec{\mathrm{v}}}{|\vec{\mathrm{r}}|^2}$ where $\vec{\mathrm{r}}$ is Q's position vector from N_o and $\vec{\mathrm{v}} \triangleq \dfrac{{}^N d\vec{\mathrm{r}}}{dt}$ is Q's velocity in N. For both situations (Q on circle and Q on horizontal line), this 3^{rd} possibility for $\vec{\omega}$ corresponds to ${}^N\vec{\omega}^A$ / ${}^N\vec{\omega}^B$ / **Neither** / **Both** (circle one).

Note: Although $\vec{\omega} = \dfrac{\vec{\mathrm{r}} \times \vec{\mathrm{v}}}{|\vec{\mathrm{r}}|^2}$ is found in textbooks and websites, it is a poor definition for angular velocity.

Related information is provided in Homework 6.27.

6.22 Optional: Angular velocity and rotation matrices. (Sections 7.3.8 and 7.3.9)

The orientation of a rigid body B in a reference frame N is specified by the following rotation table that relates the right-handed orthogonal unit vectors $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ fixed in B with the right-handed orthogonal unit vectors $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$ fixed in N. Find the $\hat{b}_y$ measure of ${}^{N}\vec{\omega}^{B}$.[a]



---

[a] If you use MotionGenesis, the `Explicit()` or `Expand()` commands simplify results.

${}^{b}R^{n}$	$\hat{n}_x$	$\hat{n}_y$	$\hat{n}_z$
$\hat{b}_x$	$\cos(q_y)\cos(q_z)$	$\sin(q_z)\cos(q_x)+\sin(q_x)\sin(q_y)\cos(q_z)$	$\sin(q_x)\sin(q_z)-\sin(q_y)\cos(q_x)\cos(q_z)$
$\hat{b}_y$	$-\sin(q_z)\cos(q_y)$	$\cos(q_x)\cos(q_z)-\sin(q_x)\sin(q_y)\sin(q_z)$	$\sin(q_x)\cos(q_z)+\sin(q_y)\sin(q_z)\cos(q_x)$
$\hat{b}_z$	$\sin(q_y)$	$-\sin(q_x)\cos(q_y)$	$\cos(q_x)\cos(q_y)$

Result:

$$^{N}\vec{\omega}^{B} = [\sin(q_z)\,\dot{q}_y + \cos(q_y)\cos(q_z)\,\dot{q}_x]\,\hat{b}_x + \qquad \hat{b}_y + [\dot{q}_z + \sin(q_y)\,\dot{q}_x]\,\hat{b}_z$$

6.23 Textbook definitions of angular velocity. (Section 7.3)

Famed dynamicist Thomas Kane called angular velocity "**one of the most misunderstood concepts in kinematics.**" Report an Internet and physics/engineering textbook definition of angular velocity and determine if the quantities appearing in the definition are **rigorously defined** - and whether they are generally applicable or only apply for simple angular velocity (described in Section 7.3.3).

Note: A definition should be able to **prove** important theorems [such as the angular velocity addition theorem of equation (7.4) and the golden rule for vector differentiation in equation (7.1)] and allow for angular velocity **calculations**.

	Definition	Rigorously defined	Works for 3D kinematics?
Internet:	Record equation/definition	Yes/No	Yes/No
Textbook:	Record equation/definition	Yes/No	Yes/No

6.24 Angular acceleration addition theorem. (Sections 7.3, 7.3.5, 7.4, 7.4.1)

Use the angular velocity addition theorem and the definition of angular acceleration to prove equation (7.10):

$$\boxed{^{N}\vec{\alpha}^{B} \underset{(7.10)}{=} {}^{N}\vec{\alpha}^{A} + {}^{A}\vec{\alpha}^{B} + {}^{N}\vec{\omega}^{A} \times {}^{A}\vec{\omega}^{B}}$$

6.25 Example of angular velocity/acceleration addition theorem (Sections 7.3.5,, 7.4.1, 7.4.1, Hw 6.24)

The following table gives the angular velocities/accelerations of A in N and B in A at a certain *instant* of time. Calculate B's angular velocity in N and B's angular acceleration in N.

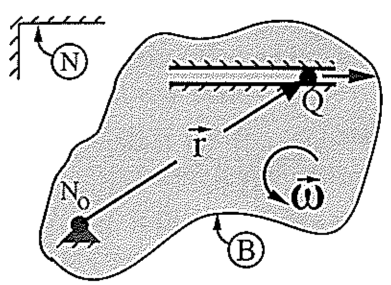
$$^{N}\vec{\omega}^{A} = 2\,\hat{a}_x \qquad {}^{N}\vec{\alpha}^{A} = 12\,\hat{a}_x$$
$$^{A}\vec{\omega}^{B} = -4\,\hat{a}_y \qquad {}^{A}\vec{\alpha}^{B} = 13\,\hat{a}_y$$
$$^{N}\vec{\omega}^{B} = 2\,\hat{a}_x - \quad \hat{a}_y \qquad {}^{N}\vec{\alpha}^{B} = 12\,\hat{a}_x + 13\,\hat{a}_y - \quad \hat{a}_z$$

6.26 Optional: Prove ${}^{B}\vec{\omega}^{A} = -{}^{A}\vec{\omega}^{B}$ and ${}^{B}\vec{\alpha}^{A} = -{}^{A}\vec{\alpha}^{B}$ for reference frames A and B

After showing ${}^{A}\vec{\omega}^{A} = \vec{0}$, use the angular velocity addition theorem to prove ${}^{B}\vec{\omega}^{A} = -{}^{A}\vec{\omega}^{B}$. Subsequently, prove ${}^{B}\vec{\alpha}^{A} = -{}^{A}\vec{\alpha}^{B}$.

6.27 Angular velocity $\vec{\omega}$ in terms of velocity $\vec{v}$ and position $\vec{r}$ (2D motion)

The figure to the right shows a rigid body B rotating in a reference frame N. Point N_o is fixed on N and is stationary (does not move) on B. Point Q moves on B. $\vec{r}$ is Q's position vector from N_o.



Knowing $\vec{v}$ (Q's velocity in N) is the time-derivative in N of $\vec{r}$, calculate $\vec{v}$ in terms of ${}^{N}\vec{\omega}^{B}$ (B's angular velocity in N), $\vec{r}$, and $\frac{^{B}d\vec{r}}{dt}$.

Result:

$$\vec{v} \triangleq \frac{^{N}d\vec{r}}{dt} =$$

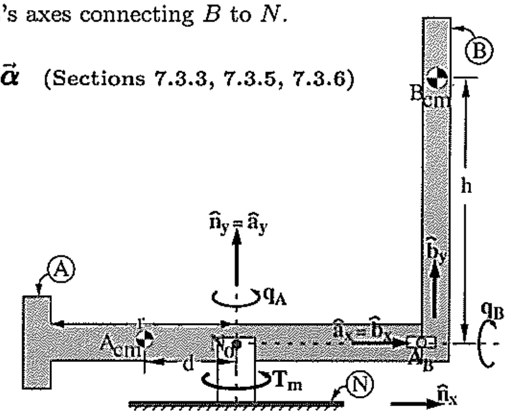
Rearrange the previous result to form an expression for ${}^{N}\vec{\omega}^{B}$ in terms of $\vec{v}$ that is valid when ${}^{N}\vec{\omega}^{B}$ is perpendicular to $\vec{r}$ (e.g., when B has a simple angular velocity in N in a plane perpendicular to $\vec{r}$).[4] Simplify this expression for the situation when Q is fixed on B.

Result:

$$Q \text{ moving on } B\text{:}\quad {}^{N}\vec{\omega}^{B} \underset{(2D)}{=} \frac{\quad \times \left(\vec{v} - \frac{^{B}d\vec{r}}{dt}\right)}{\quad \cdot \quad} \qquad Q \text{ fixed on } B\text{:}\quad {}^{N}\vec{\omega}^{B} \underset{(2D)}{=} \frac{\vec{r} \times \vec{v}}{\vec{r} \cdot \vec{r}}$$

Note: These specialized <u>2D</u> expressions for ${}^{N}\vec{\omega}^{B}$ <u>also</u> depend on P being in the plane passing through Q and perpendicular to ${}^{N}\vec{\omega}^{B}$, i.e., P is not an arbitrary point on the revolute-joint's axes connecting B to N.

6.28 Optional: Inverted pendulum on a rotating disk $\vec{\omega}$, $\vec{\alpha}$ (Sections 7.3.3, 7.3.5, 7.3.6)



The figure to the right shows a thin rigid inverted pendulum B connected to a rigid disk A by a revolute joint at point A_B. The torque motor at point N_o rotates A in a Newtonian reference frame N. Right-handed orthogonal sets of unit vectors $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$, $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$, and $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ are fixed in N, A, B, respectively, with:

- $\hat{n}_x$ horizontally-right
- $\hat{n}_y = \hat{a}_y$ vertically-upward and parallel to A's axis of rotation in N
- $\hat{a}_x = \hat{b}_x$ parallel to B's axis of rotation in A (parallel to the line connecting N_o and A_B)
- $\hat{b}_y$ directed from A_B to the distal end of B (along B's long axis)

Quantity	Symbol	Type
Angle from $\hat{n}_x$ to $\hat{a}_x$ with $+\hat{n}_y$ sense	q_A	Variable
Angle from $\hat{a}_y$ to $\hat{b}_y$ with $+\hat{a}_x$ sense (i.e., "pendulum" angle)	q_B	Variable

Determine B's angular velocity in N in terms of $\dot{q}_A$, $\dot{q}_B$, and $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$.
Determine B's angular acceleration in N in terms of $\dot{q}_A$, $\ddot{q}_A$, $\dot{q}_B$, $\ddot{q}_B$, and $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$.

Result: $^{N}\vec{\omega}^{B} = \quad \hat{a}_x + \quad \hat{a}_y \qquad {}^{N}\vec{\alpha}^{B} = \quad \hat{a}_x + \quad \hat{a}_y + \quad \hat{a}_z$

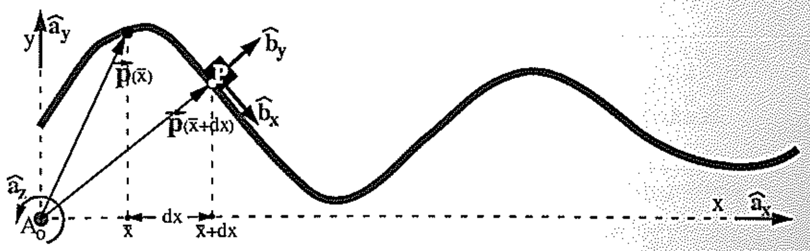
---

[4] One way to solve for $\vec{\omega}$ is to pre-cross multiply your equation for $\vec{v}$ with $\vec{r}$ and rearrange.

6.29 Optional: Unit vectors tangent and normal to a 2D-curve and the definition of angular velocity (i.e., angular velocity and curvilinear coordinates).

The figure to the right shows an arbitrary point P of a planar curve that is fixed in a reference frame A. Also shown are P's position at two values of x, namely $x = \bar{x}$ and $x = \bar{x} + dx$.

Fixed in reference frame A are a point A_o and right-handed orthogonal unit vectors $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$, with $\hat{a}_x$ horizontally-right, $\hat{a}_y$ vertically-upward, and $\hat{a}_z$ perpendicular to the planar curve.



(a) **Draw** the vector ${}^A d\vec{p} \triangleq \vec{p}(\bar{x} + dx) - \vec{p}(\bar{x})$ so its tip ends at the tip of $\vec{p}(\bar{x} + dx)$.
In the limit as $dx \to 0$, it appears ${}^A d\vec{p}$ is **tangent/normal** to the curve.

(b) P's position vector from A_o can be written in terms of scalar measures x and y as shown below.[5]
Knowing $\vec{p}$ is a vector function in A of a single scalar variable find $\frac{{}^A d\vec{p}}{ds}$ (in terms of $\frac{dx}{ds}$, $\frac{dy}{ds}$).
Result:

$$\vec{p} \underset{\text{(given)}}{=} x(s)\,\hat{a}_x + y(s)\,\hat{a}_y \qquad \Rightarrow \qquad \frac{{}^A d\vec{p}}{ds} = \qquad \hat{a}_x + \qquad \hat{a}_y$$

(c) A rigid basis B consists of right-handed orthogonal unit vectors with: $\hat{b}_x$ **tangent** to the curve (sense determined by $+dx$), $\hat{b}_y$ **normal** to the curve, and $\hat{b}_z = \hat{a}_z$. Form the ${}^bR^a$ rotation table relating $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ to $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ in terms of $\frac{dx}{ds}$, $\frac{dy}{ds}$, and r (defined below).
Optional: Calculate ${}^bR^a$ when $s = x$, $y(x) = 1 + e^{-0.1\,x}\sin(x)$, and $x = 0$.
Result:

$$r \triangleq \frac{1}{\sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}}$$

${}^bR^a$	$\hat{a}_x$	$\hat{a}_y$	$\hat{a}_z$
$\hat{b}_x$		$r\,\frac{dy}{ds}$	
$\hat{b}_y$	$-r\,\frac{dy}{ds}$		
$\hat{b}_z$			

${}^bR^a$	$\hat{a}_x$	$\hat{a}_y$	$\hat{a}_z$
$\hat{b}_x$	0.707	0.707	0
$\hat{b}_y$	-0.707	0.707	0
$\hat{b}_z$	0	0	1

(d) Show B's angular velocity in A can be expressed as given below.[6]
When P moves down a straight hill inclined at 45°, ${}^A\vec{\omega}^B = \vec{0}$ **True/False**.
Optional: Calculate ${}^A\vec{\omega}^B$ when $y(x) = 1 + e^{-0.1\,x}\sin(x)$, $x = 0$, and $\dot{x} = 1$.
Result:

$$\boxed{{}^A\vec{\omega}^B = \dot{s}\,r^2 \left(-\frac{dy}{ds}\frac{d^2x}{ds^2} + \frac{dx}{ds}\frac{d^2y}{ds^2}\right)\hat{b}_z} \qquad {}^A\vec{\omega}^B = -0.1\,\hat{b}_z$$

(e) Calculate B's angular acceleration in A in terms of r, $\dot{r}$, $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{d^2x}{ds^2}$, and $\frac{d^2y}{ds^2}$.
Result:

$${}^A\vec{\alpha}^B = \left[(\ddot{s}\,r^2 + 2\,\dot{s}\,r\,\dot{r})\left(-\frac{dy}{ds}\frac{d^2x}{ds^2} + \frac{dx}{ds}\frac{d^2y}{ds^2}\right) + \dot{s}^2 r^2 \left(-\frac{dy}{ds}\frac{d^3x}{ds^3} + \frac{dx}{ds}\frac{d^3y}{ds^3}\right)\right]\hat{b}_z$$

Calculating the ***vector tangent***, ***vector principal normal***, ***vector binormal***, and ***vector radius of curvature***, of a general 3D space curve is more complicated. The ***Serret-Frenet*** formulas for the position of a point P on a space curve as a function of the ***arc-length*** are given in [35, pg. 263] and [37, pgs. 42-47]. More general formulas for the position of P as a function of ***any*** variable are given in [37, pgs. 29-42]. More information on angular velocity, curvilinear coordinates, and differential geometry is in [42] and [38].

---

[5] x may be regarded as an independent variable or depend on another scalar variable. Hence s may stand for x, time, measure along the curve, etc.

[6] The chain rule for differentiation [equation (1.31)] is useful to show $\frac{d}{dt}\left(\frac{dx}{ds}\right) = \frac{d^2x}{ds^2}\,\dot{s}$ and $\frac{d}{dt}\left(\frac{dy}{ds}\right) = \frac{d^2y}{ds^2}\,\dot{s}$.

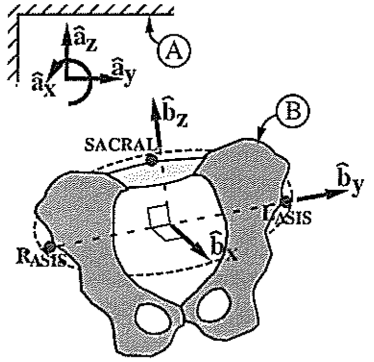
7.1 Clinical determination of pelvis orientation (described in Section 8.3).

The following figures shows two sets of right-handed orthogonal unit vectors, namely $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ fixed in a rigid body B (e.g., a pelvis) and $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ fixed in a reference frame A (e.g., a gait laboratory).

The orientation of B in A can be described *mathematically* by first setting $\hat{b}_i = \hat{a}_i$ $(i = x, y, z)$ and then subjecting B to successive right-handed rotations relative to A. Two such sequences are:

Name	Rotation sequence order		
TOR	$\theta_t\,\hat{b}_y$	$\theta_o\,\hat{b}_x$	$\theta_r\,\hat{b}_z$
ROT	$\theta_r\,\hat{b}_z$	$\theta_o\,\hat{b}_x$	$\theta_t\,\hat{b}_y$

This problem shows rotation sequence order affects the rotation matrix and shows the significant difference between mathematically-defined rotation angles (e.g., `TOR` or `ROT`) and clinically-defined angles.



(a) Form the ${}^bR^a$ rotation tables sequences.[1]

	${}^bR^a$	$\hat{a}_x$	$\hat{a}_y$	$\hat{a}_z$
TOR	$\hat{b}_x$	$\cos\theta_r\,\cos\theta_t + \sin\theta_o\,\sin\theta_r\,\sin\theta_t$	$\sin\theta_r\,\cos\theta_o$	$\sin\theta_o\,\sin\theta_r\,\cos\theta_t - \sin\theta_t\,\cos\theta_r$
	$\hat{b}_y$	$\sin\theta_o\,\sin\theta_t\,\cos\theta_r - \sin\theta_r\,\cos\theta_t$	$\cos\theta_o\,\cos\theta_r$	$\sin\theta_r\,\sin\theta_t + \sin\theta_o\,\cos\theta_r\,\cos\theta_t$
	$\hat{b}_z$	$\sin\theta_t\,\cos\theta_o$	$-\sin\theta_o$	$\cos\theta_o\,\cos\theta_t$

	${}^bR^a$	$\hat{a}_x$	$\hat{a}_y$	$\hat{a}_z$
ROT	$\hat{b}_x$	$\cos\theta_r\,\cos\theta_t - \sin\theta_o\,\sin\theta_r\,\sin\theta_t$	$\sin\theta_r\,\cos\theta_t + \sin\theta_o\,\sin\theta_t\,\cos\theta_r$	$-\sin\theta_t\,\cos\theta_o$
	$\hat{b}_y$	$-\sin\theta_r\,\cos\theta_o$	$\cos\theta_o\,\cos\theta_r$	$\sin\theta_o$
	$\hat{b}_z$	$\sin\theta_t\,\cos\theta_r + \sin\theta_o\,\sin\theta_r\,\cos\theta_t$	$\sin\theta_r\,\sin\theta_t - \sin\theta_o\,\cos\theta_r\,\cos\theta_t$	$\cos\theta_o\,\cos\theta_t$

The MotionGenesis command for `TOR` is B.Rotate(A, BodyYXZ, θ_t, θ_o, θ_r). `ROT` uses B.Rotate(A, BodyZXY, θ_r, θ_o, θ_t).

(b) ***Clinically***, the ***pelvis elevation angle*** ϕ is defined as the angle of $\hat{b}_y$ above the horizontal plane perpendicular to $\hat{a}_z$. Express ϕ in terms of θ_r, θ_o, and θ_t, first with `TOR` and then `ROT`.
Result: [Results simplify by noting $\text{acos}\,[\sin\theta_o] = \text{acos}\,[\cos(90° - \theta_o)] = 90° - \theta_o$.]

`TOR` successive-rotations: $\phi = 90° - \text{acos}\,(\sin\theta_r\,\sin\theta_t + \sin\theta_o\,\cos\theta_r\,\cos\theta_t)$

`ROT` successive-rotations: $\phi = 90° - \text{acos}\,(\sin\theta_o) \quad (= \theta_o \text{ when } -90° \le \theta_o \le 90)$

(c) ***Clinically***, the ***pelvis progression angle*** ψ is defined as the angle of $\hat{b}_y$ behind the vertical plane perpendicular to $\hat{a}_x$. Express ψ in terms of θ_r, θ_o, and θ_t, first with `TOR` and then `ROT`.
Result: [Results simplify by noting $\text{acos}(-x) = 180° - \text{acos}(x)$.]

`TOR` successive-rotations: $\psi = 90° - \text{acos}\,(\sin\theta_r\,\cos\theta_t - \sin\theta_o\,\sin\theta_t\,\cos\theta_r)$

`ROT` successive-rotations: $\psi = 90° - \text{acos}\,(\sin\theta_r\,\cos\theta_o)$

(d) ***Clinically***, the ***pelvis lean angle*** γ is defined as the angle of $\hat{b}_x$ below the horizontal plane perpendicular to $\hat{a}_z$. Express γ in terms of θ_r, θ_o, and θ_t, first by with `TOR` and then `ROT`.
Result:

`TOR` successive-rotations: $\gamma = 90° - \text{acos}\,(\sin\theta_t\,\cos\theta_r - \sin\theta_o\,\sin\theta_r\,\cos\theta_t)$

`ROT` successive-rotations: $\gamma = 90° - \text{acos}\,(\sin\theta_t\,\cos\theta_o)$

---

[1] Reference: Wren, Tishya, and Mitiguy, Paul, "A Simple Method to Obtain Consistent and Clinically Meaningful Pelvic Angles from Euler Angles during Gait Analysis", Journal of Applied Biomechanics. Vol. 23, No. 3, 2007, pp. 28-223.