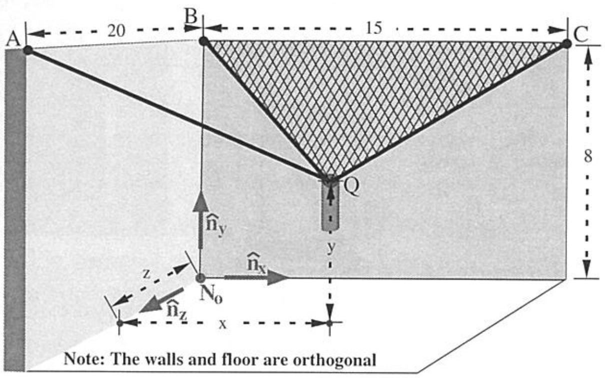
## 3.1 Position concepts - what objects have a position vector?

**Draw** $\vec{r}^{\,S/O}$, the position vector of an object $S$ from a point $O$.
In general and **without ambiguity**, $S$ could be a (circle all appropriate objects):

| | | | |
|---|---|---|---|
| Scalar | Real number | Complex number | Center of a circle |
| Vector | Point | Reference Frame | Mass center of a set of particles |
| Matrix | Set of Points | Rigid Body | Mass center of a rigid body |
| Line | Particle | Flexible Body | Set of flexible bodies |
| Orthogonal unit basis | Set of Particles | Set of Rigid bodies | System of particles and bodies |

## 3.2 Microphone cable geometry: Lengths, angles, and surface area/normal.

A microphone $Q$ is attached to three pegs $A$, $B$, and $C$ by three cables.[1] [2] [3]




Peg and microphone locations from point $N_o$ are known

| Quantity | Symbol | Value |
|---|---|---|
| Distance from $A$ to $B$ | $AB$ | 20 m |
| Distance from $B$ to $C$ | $BC$ | 15 m |
| Distance from $N_o$ to $B$ | $h$ | 8 m |
| $\hat{n}_x$ measure of $\vec{r}^{\,Q/N_o}$ | $x$ | 7 m |
| $\hat{n}_y$ measure of $\vec{r}^{\,Q/N_o}$ | $y$ | 5 m |
| $\hat{n}_z$ measure of $\vec{r}^{\,Q/N_o}$ | $z$ | 8 m |

$$\vec{r}^{\,Q/N_o} = x\,\hat{n}_x + y\,\hat{n}_y + z\,\hat{n}_z$$

Calculate the following values ($2^+$ significant digits)

| Quantity | Symbol | Value |
|---|---|---|
| Length of cable joining $A$ and $Q$ | $L_A$ | 14.2 m |
| Length of cable joining $B$ and $Q$ | $L_B$ | m |
| Length of cable joining $C$ and $Q$ | $L_C$ | m |
| Angle $\phi$ between line $\overline{BQ}$ and line $\overline{BC}$ | $\phi$ | 50.7 ° |
| Mass of thin cloth (density 0.1 kg/m$^2$) whose surface area covers ΔBCQ | $m$ | 6.41 kg |
| Unit vector perpendicular to ΔBCQ | $\hat{u}$ | 0.936 $\hat{n}_y$ + $\hat{n}_z$ |

Note: ΔBCQ is the triangle formed by points $B$, $C$, and $Q$.

[1] Section 3.4 shows how to determine the area of the triangle formed by points $A$, $B$, and $Q$.
[2] Section 3.4 shows how to determine a unit vector perpendicular to the triangle formed by points $A$, $B$, and $Q$.
[3] Section 3.3 shows how to determine the angle between line $\overline{AQ}$ and $\overline{AB}$.

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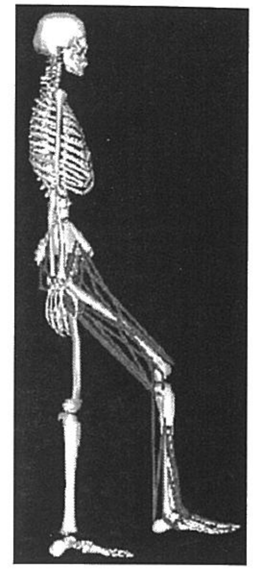
## 3.3 Ankle angle and anatomical landmarks

The following human lower-leg schematic has the relevant anatomical bony landmarks:

- Point $A$: Midpoint of the lateral and medial ***malleoli*** (near the ankle)
- Point $B$: Near the ***metatarsal phalangeal*** (towards the big toe)
- Point $C$: Midpoint of lateral and medial ***femoral condyles*** (near the knee)

Reflective markers are attached to points $A$, $B$, and $C$ to capture their positions from a point $N_o$ (fixed in a motion-capture laboratory $N$). ***Motion capture*** hardware and software record the following marker data in terms of orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ fixed in $N$.

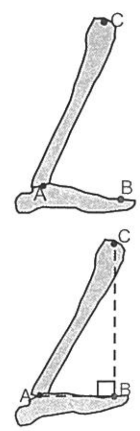


Courtesy of Drs. Dan Jacobs & Gabriel Sanchez

$$\vec{\mathbf{r}}^{\,A/N_o} = x_A\,\hat{\mathbf{n}}_x + y_A\,\hat{\mathbf{n}}_y + z_A\,\hat{\mathbf{n}}_z$$

$$\vec{\mathbf{r}}^{\,B/N_o} = x_B\,\hat{\mathbf{n}}_x + y_B\,\hat{\mathbf{n}}_y + z_B\,\hat{\mathbf{n}}_z$$

$$\vec{\mathbf{r}}^{\,C/N_o} = x_C\,\hat{\mathbf{n}}_x + y_C\,\hat{\mathbf{n}}_y + z_C\,\hat{\mathbf{n}}_z$$




Determine the value of the "***ankle angle***" $\theta$ between the vector from $A$ to $B$ and the vector from $A$ to $C$ for the following special-case of a **right triangle**.

Note: This result is useful for **verifying** the more general expression.

$x_A = 0$ $\quad y_A = 0$ $\quad z_A = 0$

$x_B = 1$ $\quad y_B = 0$ $\quad z_B = 0$ $\qquad$ **Result:** $\theta =$ ____ $^\circ$

$x_C = 1$ $\quad y_C = 1$ $\quad z_C = 0$

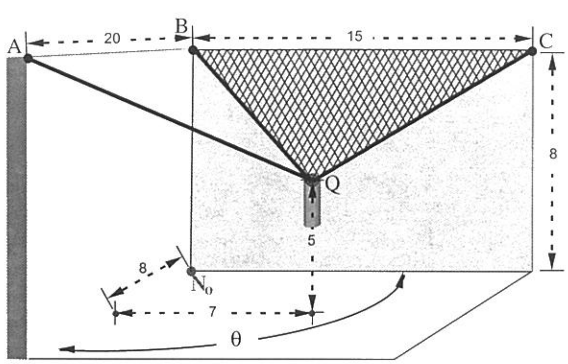
Find a general expression for $\theta$ in terms of $x_A, y_A, z_A,\ x_B, y_B, z_B,\ x_C, y_C, z_C$.

**Result:**

$$\theta = \qquad \left\{ \frac{\quad + \quad + \quad}{\sqrt{\quad + \quad + \quad}\ \sqrt{\quad + \quad + \quad}} \right\}$$

## 3.4 Microphone cable surface area (non-orthogonal walls).

A microphone $Q$ is attached to three pegs $A$, $B$, and $C$ by three cables. Knowing the peg locations, microphone location, express the surface area of the cloth that covers the triangle formed by points $B$, $C$, $Q$ in terms of the angle $\theta$ between the vertical walls.




| Quantity | Value |
|---|---|
| Distance from $A$ to $B$ | 20 m |
| Distance from $B$ to $C$ | 15 m |
| Distance from $N_o$ to $B$ | 8 m |
| $Q$'s measure from $N_o$ along $\overline{BC}$ | 7 m |
| $Q$'s height above $N_o$ | 5 m |
| $Q$'s measure from $N_o$ along $\overline{BA}$ | 8 m |

**Result:**

Surface area $= \sqrt{\quad^2 + 120^2 \sin^2(\theta)}$

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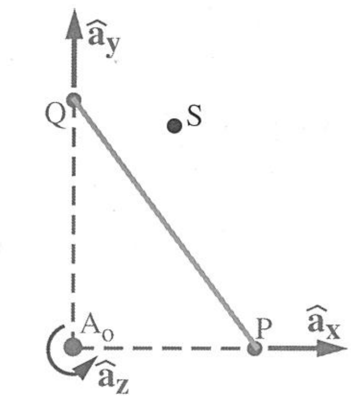
### 3.5 † Distance from a line (2D/3D geometry)

A common geometry problem is to determine how far a point is from a line. The figure to the right shows a line $\overline{PQ}$ passing through points $P$ and $Q$. The position vectors of $P$ and $Q$ from a point $A_o$ are written in terms of right-handed, orthogonal, unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$, as

$$\vec{\mathbf{r}}^{\,P/A_o} = p_x\,\widehat{\mathbf{a}}_x \qquad\qquad \vec{\mathbf{r}}^{\,Q/A_o} = q_y\,\widehat{\mathbf{a}}_y$$




A point $S$ is located so that: $\vec{\mathbf{r}}^{\,S/A_o} = s_x\,\widehat{\mathbf{a}}_x + s_y\,\widehat{\mathbf{a}}_y + s_z\,\widehat{\mathbf{a}}_z$.

Denoting $C$ as the point on line $\overline{PQ}$ closest to $S$, determine the following (in terms of $p_x$, $q_y$, $s_x$, $s_y$, $s_z$ and symbols in the table below).

| | | |
|---|---|---|
| Distance between $P$ and $Q$ | $d^{Q/P}$ | $= \sqrt{p_x^2 + q_y^2}$ |
| Distance between $P$ and $C$ | $d^{P/C}$ | $= \dfrac{\rule{3cm}{0.4pt}}{d^{Q/P}}$ |
| $C$'s position vector from $P$ | $\vec{\mathbf{r}}^{\,C/P}$ | $= \rule{1.5cm}{0.4pt}\ \widehat{\mathbf{a}}_x + \rule{1.5cm}{0.4pt}\ \widehat{\mathbf{a}}_y$ |

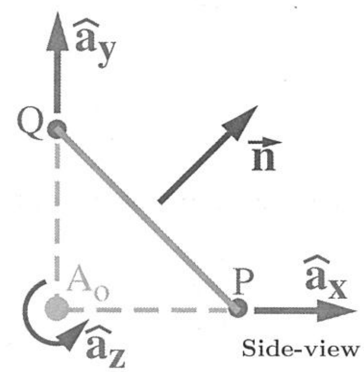
Using $p_x = q_y = s_x = 1$ m, $s_y = s_z = 0.5$ m, calculate the distance between $S$ and $C$.

**Result:** ($2^+$ significant digits) $d^{S/C} = \boxed{0.6123724}$ m

### 3.6 Distance from a plane (2D geometry)

A common geometry problem is to determine how far a point is from a plane. For example, the following figure shows an $\widehat{\mathbf{a}}_z$ side-view of a rectangular skylight window with side-view corners at points $P$ and $Q$, whose position vectors from $A_o$ (a point inside the window/building) are written in terms of right-handed, orthogonal, unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$, as

$$\vec{\mathbf{r}}^{\,P/A_o} = 1\text{ m}\ \widehat{\mathbf{a}}_x \qquad\qquad \vec{\mathbf{r}}^{\,Q/A_o} = 1\text{ m}\ \widehat{\mathbf{a}}_y$$




A street-lamp (not shown) is planned to be built near the window with a point $S$ (the point of the street-lamp nearest the window) to be located at

$$\vec{\mathbf{r}}^{\,S/A_o} = 1\text{ m}\,\widehat{\mathbf{a}}_x + 0.5\text{ m}\,\widehat{\mathbf{a}}_y$$

Calculate (or guess) a vector $\vec{\mathbf{n}}$ outward-normal to the skylight window.

**Result:** ($\vec{\mathbf{n}}$ is perpendicular to the window and points outside.)

$\vec{\mathbf{n}} = \underline{\quad}\ \widehat{\mathbf{a}}_x + 1\,\widehat{\mathbf{a}}_y$ ($\vec{\mathbf{n}}$ is <u>not</u> a unit vector since $|\vec{\mathbf{n}}| = \sqrt{2}$)

Determine $d$, how far $S$ is outside the infinite plane containing the rectangular skylight window.

**Result:** (A positive value of $d$ means $S$ is outside the building whereas a negative $d$ means $S$ is inside.)

$d =$ meters

† Leaving all data unchanged except to modifying the angle between $\widehat{\mathbf{a}}_x$ and $\widehat{\mathbf{a}}_y$ to 60°, cleverly redo this problem only using the now **non-orthogonal**, basis, $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$.

**Result:**

$\vec{\mathbf{n}} = \widehat{\mathbf{a}}_z \times (\quad \widehat{\mathbf{a}}_x - 1\,\widehat{\mathbf{a}}_y) \qquad |\vec{\mathbf{n}}| = 1 \qquad d = \dfrac{\vec{\mathbf{n}}}{|\vec{\mathbf{n}}|} \cdot \qquad = 0.5\sin(60)^\circ \approx 0.43$ meters

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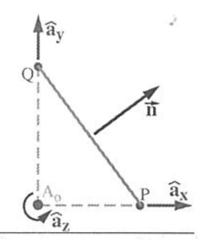


The problem statement can be generalized using symbols $p_x$, $q_y$, $s_x$, $s_y$ where

$$\vec{\mathbf{r}}^{\,P/A_o} = p_x\,\widehat{\mathbf{a}}_x \qquad \vec{\mathbf{r}}^{\,Q/A_o} = q_y\,\widehat{\mathbf{a}}_y \qquad \vec{\mathbf{r}}^{\,S/A_o} = s_x\,\widehat{\mathbf{a}}_x + s_y\,\widehat{\mathbf{a}}_y$$

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Calculate a **unit** vector $\hat{\mathbf{n}}$ outward-normal to the skylight window. Express $d$, how far $S$ is outside the infinite plane containing the rectangular skylight window, in terms of $p_x$, $q_y$, $s_x$, $s_y$.

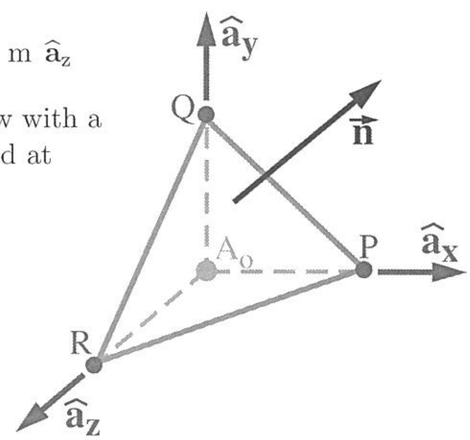
**Result:** ($\hat{\mathbf{n}}$ is perpendicular to the window and points outside and $d$ is positive if $S$ is outside the building).

$$\hat{\mathbf{n}} = \_\_\_\_\_\_ \hat{\mathbf{a}}_x + \_\_\_\_\_\_ \hat{\mathbf{a}}_y \qquad d = \_\_\_\_\_\_$$

### 3.7 † Distance from a building: It's "just" geometry (3D) – Courtesy Adam Leeper

A common geometry problem (arising in computer graphics, construction, biomechanics, engineering, etc.) is to determine how far a point is from a plane. For example, the following figure shows points $P$, $Q$, and $R$ located at corners of a triangular skylight window with position vectors from $A_o$ (a point inside the window/building) expressed in terms of right-handed, orthogonal, unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$, as

$$\vec{\mathbf{r}}^{P/A_o} = 1\text{ m }\hat{\mathbf{a}}_x \qquad \vec{\mathbf{r}}^{Q/A_o} = 1\text{ m }\hat{\mathbf{a}}_y \qquad \vec{\mathbf{r}}^{R/A_o} = 1\text{ m }\hat{\mathbf{a}}_z$$




A street-lamp (not shown) is planned to be built near the window with a point $S$ (the point of the street-lamp nearest the window) to be located at

$$\vec{\mathbf{r}}^{S/A_o} = 0.5\text{ m }\hat{\mathbf{a}}_x + 0.5\text{ m }\hat{\mathbf{a}}_y + 0.25\text{ m }\hat{\mathbf{a}}_z$$

Calculate a vector $\vec{\mathbf{n}}$ outward-normal to the skylight window.

**Result:** ($\vec{\mathbf{n}}$ is **not** a unit vector)

$$\vec{\mathbf{n}} = \_\_\_\_ \hat{\mathbf{a}}_x + \_\_\_\_ \hat{\mathbf{a}}_y + 1\,\hat{\mathbf{a}}_z$$

Determine the measure $d$ of how far $S$ is outside the infinite plane containing the triangular skylight window (in meters).

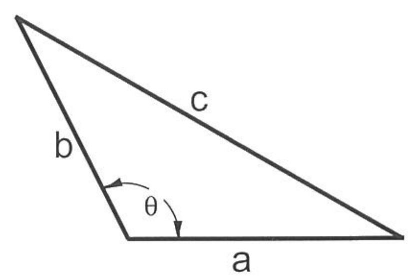
**Result:** (A positive value of $d$ means $S$ is outside the building whereas a negative $d$ means $S$ is inside.)

$$d = \qquad \approx 0.144 \text{ meters}$$

### 3.8 Proof of the law of cosines with vectors.

**Vectors** are very useful for geometry. Use **vectors** and **vector operations** to prove the ***law of cosines*** for a triangle with sides of length $a$, $b$, and $c$ and an angle $\theta$ opposite the side of length $c$.[a]

$$c^2 = a^2 + b^2 - 2\,a\,b\cos(\theta)$$




[a] Start by introducing vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ so that $\theta$ is the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.