6.12 Optional: Proofs

Proof of vector dot-product derivative property

The proof of the first equation in Section 6.6, i.e., the ***vector dot-product derivative property*** starts by expressing the arbitrary vectors $\vec{u}$ and $\vec{v}$ in terms of an arbitrary set of orthogonal unit vectors $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ fixed in an arbitrary reference frame (or rigid basis) A as[4]

$$\vec{u} = u_x\,\hat{a}_x + u_y\,\hat{a}_y + u_z\,\hat{a}_z \qquad\qquad \vec{v} = v_x\,\hat{a}_x + v_y\,\hat{a}_y + v_z\,\hat{a}_z \tag{13}$$

Forming the scalar quantity $\vec{u}\cdot\vec{v}$ and differentiating with respect to the scalar variable t, one finds

$$\begin{aligned} \vec{u}\cdot\vec{v} &\underset{(13)}{=} u_x\,v_x + u_y\,v_y + u_z\,v_z \\ \frac{d\,(\vec{u}\cdot\vec{v})}{dt} &= \dot{u}_x\,v_x + u_x\,\dot{v}_x + \dot{u}_y\,v_y + u_y\,\dot{v}_y + \dot{u}_z\,v_z + u_z\,\dot{v}_z \end{aligned} \tag{14}$$

The next step is to form the right-hand side of the equation being proved, i.e., $\frac{{}^A d\vec{u}}{dt}\cdot\vec{v} + \vec{u}\cdot\frac{{}^A d\vec{v}}{dt}$, as

$$\begin{aligned} \frac{{}^A d\vec{u}}{dt} &\underset{(13)}{\triangleq} \dot{u}_x\,\hat{a}_x + \dot{u}_y\,\hat{a}_y + \dot{u}_z\,\hat{a}_z & \qquad \frac{{}^A d\vec{u}}{dt}\cdot\vec{v} &\underset{(13)}{=} \dot{u}_x\,v_x + \dot{u}_y\,v_y + \dot{u}_z\,v_z \\ \frac{{}^A d\vec{v}}{dt} &\underset{(13)}{\triangleq} \dot{v}_x\,\hat{a}_x + \dot{v}_y\,\hat{a}_y + \dot{v}_z\,\hat{a}_z & \qquad \vec{u}\cdot\frac{{}^A d\vec{v}}{dt} &\underset{(13)}{=} u_x\,\dot{v}_x + u_y\,\dot{v}_y + u_z\,\dot{v}_z \end{aligned} \tag{15}$$

Combining the two right-most equations in the previous set of equations gives

$$\frac{{}^A d\vec{u}}{dt}\cdot\vec{v} + \vec{u}\cdot\frac{{}^A d\vec{v}}{dt} \underset{(15)}{=} \dot{u}_x\,v_x + \dot{u}_y\,v_y + \dot{u}_z\,v_z + u_x\,\dot{v}_x + u_y\,\dot{v}_y + u_z\,\dot{v}_z \tag{16}$$

This proof concludes by viewing the equivalence displayed in equations (14) and (16), and recalling that $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ are arbitrary vectors fixed in an arbitrary reference frame (or rigid basis) A. Hence, for arbitrary reference frames (or rigid bases) A and/or B,

$$\boxed{\frac{d(\vec{u}\cdot\vec{v})}{dt} = \frac{{}^A d\vec{u}}{dt}\cdot\vec{v} + \vec{u}\cdot\frac{{}^A d\vec{v}}{dt} = \frac{{}^B d\vec{u}}{dt}\cdot\vec{v} + \vec{u}\cdot\frac{{}^B d\vec{v}}{dt}}$$

Note: The ***vector dot-product derivative property*** is used to prove the uniqueness and existence of the ***golden rule for vector differentiation*** [equation (7.1)] in Section 7.5.1.

[4] The proof of the ***vector dot-product derivative property*** in Section 6.12 does not require, but is greatly simplified, with orthogonal unit vectors. To do the same proof with non-orthogonal unit vectors $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$, one must write $\vec{u}\cdot\vec{v}$ in terms of $\hat{a}_1\cdot\hat{a}_2$, $\hat{a}_1\cdot\hat{a}_3$, and $\hat{a}_2\cdot\hat{a}_3$ (which are the cosines of angles between the unit vectors) and then note that for a **rigid basis** A, the angles between these unit vectors are constant - and hence $\frac{d\,(\hat{a}_i\cdot\hat{a}_j)}{dt} = 0 \;\; (i,j=1,2,3)$.

Chapter 7

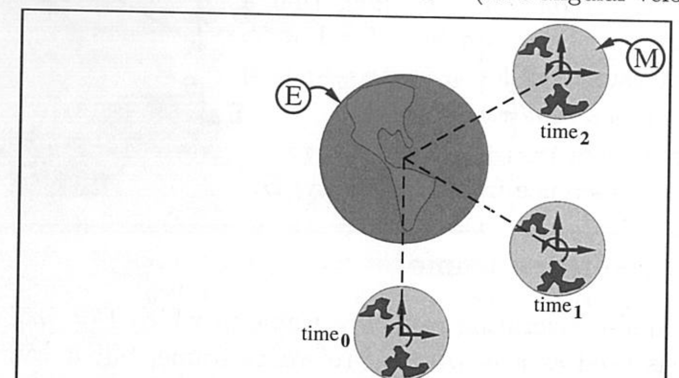
Angular velocity & angular acceleration

Important formulas for angular velocity and angular acceleration. (See examples in Hw 6)

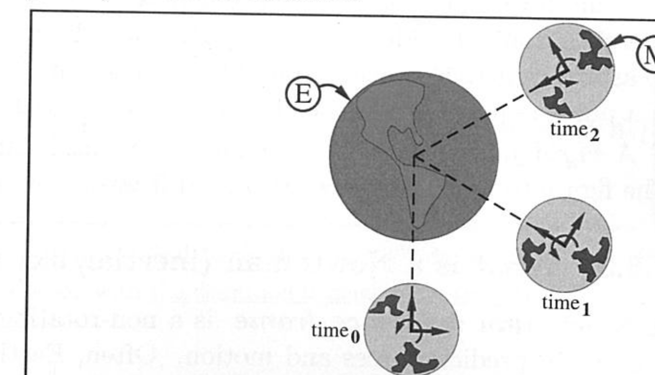
Golden rule for vector differentiation	$\frac{{}^A d\vec{v}}{dt} \underset{(7.1)}{=} \frac{{}^B d\vec{v}}{dt} + {}^A\vec{\omega}^B \times \vec{v}$
Simple angular velocity	${}^A\vec{\omega}^B \underset{(7.2)}{=} \pm\dot{\theta}\hat{\lambda}$ when $\hat{\lambda}$ is <u>fixed</u> in both A and B
Angular velocity negative property	${}^A\vec{\omega}^B \underset{(7.3)}{=} -{}^B\vec{\omega}^A$
Angular velocity addition theorem	${}^A\vec{\omega}^D \underset{(7.4)}{=} {}^A\vec{\omega}^B + {}^B\vec{\omega}^C + {}^C\vec{\omega}^D$
Angular velocity and arbitrary vectors	See equation (5)
Angular velocity and basis vectors	${}^A\vec{\omega}^B \underset{(7.6)}{=} (\frac{{}^A d\hat{b}_y}{dt}\cdot\hat{b}_z)\,\hat{b}_x + (\frac{{}^A d\hat{b}_z}{dt}\cdot\hat{b}_x)\,\hat{b}_y + (\frac{{}^A d\hat{b}_x}{dt}\cdot\hat{b}_y)\,\hat{b}_z$
Angular velocity and rotation matrices	See equation (7)
Partial angular velocity (Kane's method)	See Section 25.2 (also for virtual angular displacement)
B's angular acceleration in A	${}^A\vec{\alpha}^B \underset{(7.8)}{\triangleq} \frac{{}^A d\,{}^A\vec{\omega}^B}{dt} \underset{(7.9)}{=} \frac{{}^B d\,{}^A\vec{\omega}^B}{dt}$
Angular acceleration addition theorem	${}^A\vec{\alpha}^C \underset{(7.10)}{=} {}^A\vec{\alpha}^B + {}^B\vec{\alpha}^C + {}^A\vec{\omega}^B \times {}^B\vec{\omega}^C$
Angular acceleration negative property	${}^A\vec{\alpha}^B \underset{(7.11)}{=} -{}^B\vec{\alpha}^A$

7.1 Angular velocity concepts: Moon and Earth celestial systems

Each of the two pictures below depict a moon M in counter-clockwise orbit about a planet E. From the pictures, one can see whether ${}^E\vec{\omega}^M$ (M's angular velocity in E) is zero or non-zero.[1]



${}^E\vec{\omega}^M \boxed{=} \vec{0}$ (M's orientation in E is <u>constant</u>.)



${}^E\vec{\omega}^M \boxed{\neq} \vec{0}$ (M's orientation in E <u>changes</u>.)

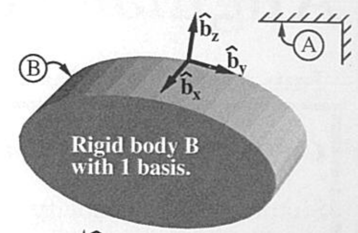
[1] A particle only <u>translates</u> whereas a rigid body can <u>translate</u> and <u>rotate</u>. A particle can translate around Earth - but the particle is not "rotating" (particles do not possess orientation). Conversely, a rigid body can translate around Earth and its orientation may or may not change. Although the left-moon's angular velocity ${}^E\vec{\omega}^M = \vec{0}$, one can construct a rigid vector basis B so ${}^E\vec{\omega}^B \neq \vec{0}$ by using an Earth-to-Moon pointing vector and a vector perpendicular to the orbital plane.

7.2 What is a reference frame?

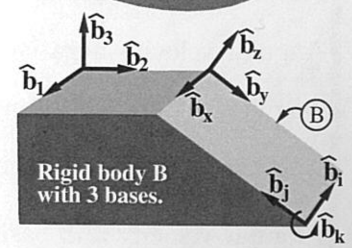
A ***reference frame*** is a rigid 3D object in which points, curves, surfaces, unit vectors, rigid vector bases, rigid frames, rigid bodies, and other rigid objects can be <u>fixed</u>. Both translational motion (e.g., a point's ***velocity*** or ***acceleration***) and rotational motion (e.g., a rigid body's ***angular velocity*** or ***angular acceleration***) are measured ***with respect to (in)*** a reference frame.

Examples of rigid objects that may be regarded as a reference frame: <u>rigid body</u>, rectangle, circle, plate, parallelogram, shell, box, cylinder, and sphere. It can be helpful to construct non-physical (non-material object) reference frames whose only purpose is to be a convenient place for fixing points, vector bases, curves, etc.

7.2.1 Orientation of a rigid body (or reference frame)



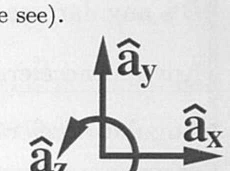
A rigid body B's orientation characterizes its alignment relative to a vector basis (or other rigid body or reference frame). Orientation may be measured by inscribing three non-coplanar unit vectors on B that are ordered into a right-handed set (i.e., a right-handed, unitary ***rigid vector basis*** fixed in B).



Although B's orientation relative to a reference frame A is unique, its measurement is not because of the freedom to fix bases on B in different ways. Sometimes it is desirable to fix multiple bases on the same rigid body. For example, the top-figure shows one basis whereas the bottom-figure shows three bases. The measurement of B's orientation in A requires extra equipment, e.g., a ***rigid vector basis*** fixed in B and a ***rigid vector basis*** fixed in A.

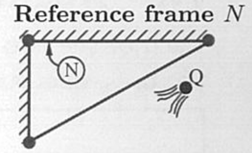
Note: Reference frames, rigid bodies, and rigid bases have orientation. ***Points and particles do <u>not</u> have orientation.***

7.2.2 Differences between a vector basis, rigid frame, and reference frame



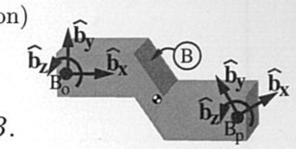
- A ***vector basis*** may span 1D, 2D, 3D, ..., space (e.g., Euclidean space – the 3D world we see). Although a vector basis implies space, it does not imply time.
- A 3D ***rigid vector basis*** (e.g., an orthogonal set of unit vectors) is special in that the angles between the basis vectors are <u>constant</u>. From "rigid" and "constant", we infer that a rigid basis must include a sense of time. With two 3D rigid vector bases, one may measure relative orientation, angular velocity, or angular acceleration.
- A ***reference frame*** is a **rigid 3D object**, defined as an infinite set of non-collinear points whose distance between each pair is <u>constant</u>. A reference frame N can be constructed by as few as three non-collinear points whose distances from each other are time invariant (**constant**). From "distance" and "rigid/constant", we infer that a reference frame includes both space and time. Using a point and vector basis fixed in N, an arbitrary (possibly moving) point Q's position can be measured. Velocity (and acceleration) of a point requires a reference frame (this cannot be done with a ***rigid vector basis***).

Reference frame N



- A ***rigid frame*** is an orthogonal unitary basis and a point ("origin").

The figure to the right shows two rigid frames fixed to a reference frame (rigid body) B.



7.2.3 What is a Newtonian (inertial/fixed) reference frame?

A ***Newtonian reference frame*** is a non-rotating, non-accelerating reference frame in which $\vec{F} = m\,\vec{a}$ accurately predicts forces and motion. Often, Earth is used as a Newtonian reference frame, but it can prove inadequate for predicting certain forces or motions, particularly for celestial bodies or long-term motion (hours$^+$). Perhaps perplexing and confounding, the rules are as follows:

- A Newtonian reference frame is a reference frame in which $\vec{F} = m\,\vec{a}$ applies.
- $\vec{F} = m\,\vec{a}$ only applies in a Newtonian reference frame.

7.3 Angular velocity and the golden rule for vector differentiation

The angular velocity ${}^{A}\vec{\omega}^{B}$ characterizes the time-rate of change of orientation of reference frame (or rigid vector basis) B in reference frame (or rigid vector basis) A. ${}^{A}\vec{\omega}^{B}$ has the defining property (called the ***golden rule for vector differentiation***) that for ***any vector*** $\vec{v}$, relates time-derivatives of $\vec{v}$ in A and B as

$$\boxed{\frac{{}^{A}d\,\text{any}\vec{\text{Vector}}}{dt} = \frac{{}^{B}d\,\text{any}\vec{\text{Vector}}}{dt} + {}^{A}\vec{\omega}^{B} \times \text{any}\vec{\text{Vector}}} \quad \text{i.e.,} \quad \boxed{\frac{{}^{A}d\vec{v}}{dt} = \frac{{}^{B}d\vec{v}}{dt} + {}^{A}\vec{\omega}^{B} \times \vec{v}} \qquad (1)$$

Reference frames, rigid bodies, and rigid vector bases have angular velocity. ***Points and particles do <u>not</u>.***

Equation (1) is also called the "***Transport theorem for vector differentiation***".

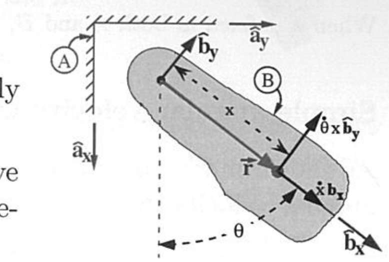
The vector $\vec{v}$ is **arbitrary**, e.g., a unit vector, position or velocity vector, or translational/angular momentum vector.

Angular velocity is defined by its property in equation (1). This is analogous to an identity matrix I whose defining property is $I * X = X * I = X$ (where X is *any* matrix), and analogous to a zero matrix whose defining property is $0 + X = X$. Similarly, the unit dyadic $\bar{\bar{1}}$ is defined in Section 15.1 by its property $\bar{\bar{1}} \cdot \vec{v} = \vec{v} \cdot \bar{\bar{1}} = \vec{v}$ (where $\vec{v}$ is *any* vector).

Although equation (1) defines angular velocity, one cannot directly solve for ${}^{A}\vec{\omega}^{B}$ in equation (1) with a single vector $\vec{v}$ and its time-derivatives in A and B. However, ${}^{A}\vec{\omega}^{B}$ is uniquely defined by equation (1) since equation (1) is valid for **any vector** $\vec{v}$. A unique expression for ${}^{A}\vec{\omega}^{B}$ can be found via **two non-parallel vectors** (e.g., $\vec{p}$ and $\vec{q}$) and their time-derivatives in A and B. For more information, see the uniqueness proof in Section 7.5.1 and equation (5) and its subsequent comments.

Although Euler discovered most of the governing properties of angular velocity, the first appearance of equation (1) with the explicit display of A and B in derivatives and angular velocity was by Thomas Kane in 1950. Until then, the equation was written in a form where the "flowing derivative" on the right was written with a δ or D to distinguish it (in a manner hard to comprehend) from the derivative on the left. The name "***golden rule for vector differentiation***" and the use of equation (1) as a ***definition*** for angular velocity were invented by the author, students, and colleagues circa 1995. The existence and uniqueness proof in Section 7.5.1 was created by the author in 2012.

7.3.1 Example: Angular velocity and vector differentiation



The ***golden rule for vector differentiation*** in equation (1) is usually <u>more efficient</u> for calculating vector derivatives than definition (6.3).

For example, Section 6.7 used definition (6.3) to calculate the time-derivative of vector $\vec{r} = x\,\hat{b}_x$ in reference frame A. Knowing ${}^{A}\vec{\omega}^{B} = \dot{\theta}\,\hat{b}_z$, the time-derivative of $\vec{r}$ in A is calculated more efficiently as

Fast: Golden rule

$$\begin{aligned}\frac{{}^{A}d\vec{r}}{dt} &\underset{(1)}{=} \frac{{}^{B}d\vec{r}}{dt} + {}^{A}\vec{\omega}^{B} \times \vec{r} \\ &= \dot{x}\,\hat{b}_x + (\dot{\theta}\,\hat{b}_z) \times (x\,\hat{b}_x) \\ &= \dot{x}\,\hat{b}_x + x\,\dot{\theta}\,\hat{b}_y\end{aligned}$$

Slow: Definition of vector derivative

$$\begin{aligned}\vec{r} &\underset{(6.5)}{=} x\,[\cos(\theta)\,\hat{a}_x + \sin(\theta)\,\hat{a}_y] \\ \frac{{}^{A}d\vec{r}}{dt} &\underset{(6.3)}{=} \dot{x}\,[\cos(\theta)\,\hat{a}_x + \sin(\theta)\,\hat{a}_y] + x\,\dot{\theta}\,[-\sin(\theta)\,\hat{a}_x + \cos(\theta)\,\hat{a}_y] \\ &\underset{(6.5)}{=} \dot{x}\,\hat{b}_x + x\,\dot{\theta}\,\hat{b}_y\end{aligned}$$

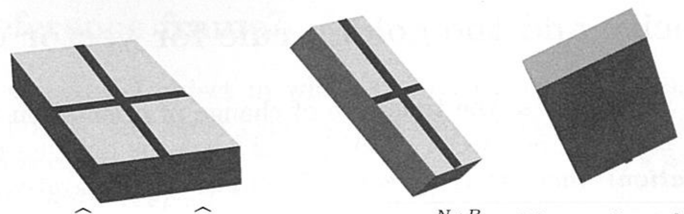
The ***golden rule for vector differentiation*** makes it <u>easy</u> to also calculate acceleration (2^{nd}-derivative of $\vec{r}$ in A) as $(\ddot{x} - x\,\dot{\theta}^2)\,\hat{b}_x + (2\,\dot{x}\,\dot{\theta}\,\hat{b}_y + x\,\ddot{\theta})\,\hat{b}_y$, whereas calculating acceleration with <u>the definition is much more tedious</u>.

7.3.2 Example: Vector differentiation and angular momentum – spinning book

The angular momentum principle for a rigid body B in a Newtonian reference frame N is

$$\vec{M} = \frac{{}^{N}d\,\vec{H}}{dt}$$

$\vec{M}$ is the moment of all forces on B about B_{cm} (B's center of mass)
$\vec{H}$ is the angular momentum in N of B about B_{cm}



Given: $\quad {}^{N}\vec{\omega}^{B} = \omega_x\,\hat{\mathbf{b}}_x + \omega_y\,\hat{\mathbf{b}}_y + \omega_z\,\hat{\mathbf{b}}_z \qquad {}^{N}\vec{\omega}^{B}$ is B's angular velocity in N

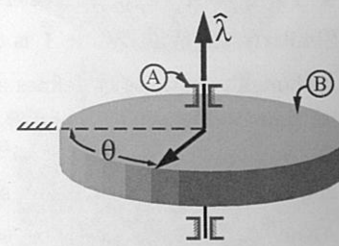
$\vec{\mathrm{H}} = \mathrm{I}_{xx}\,\omega_x\,\hat{\mathbf{b}}_x + \mathrm{I}_{yy}\,\omega_y\,\hat{\mathbf{b}}_y + \mathrm{I}_{zz}\,\omega_z\,\hat{\mathbf{b}}_z \qquad \mathrm{I}_{xx}, \mathrm{I}_{yy}, \mathrm{I}_{zz}$ are constants (moments of inertia)

$\vec{\mathrm{M}} = \vec{0}$ No external forces

where ω_x, ω_y, ω_z are time-dependent variables and $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ are right-handed orthogonal unit vectors fixed in B. Differentiating $\vec{\mathrm{H}}$ in N gives the following equation that governs B's rotational motion.

$$[\mathrm{I}_{xx}\,\dot{\omega}_x + (\mathrm{I}_{zz}-\mathrm{I}_{yy})\,\omega_z\,\omega_y]\,\hat{\mathbf{b}}_x \;+\; [\mathrm{I}_{yy}\,\dot{\omega}_y + (\mathrm{I}_{xx}-\mathrm{I}_{zz})\,\omega_x\,\omega_z]\,\hat{\mathbf{b}}_y \;+\; [\boxed{\mathrm{I}_{zz}\,\dot{\omega}_z + (\mathrm{I}_{yy}-\mathrm{I}_{xx})\,\omega_y\,\omega_x}]\,\hat{\mathbf{b}}_z = \vec{0}$$

7.3.3 Simple angular velocity



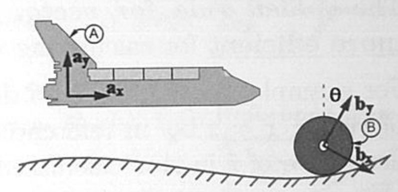
Consider two reference frames (or rigid vector bases or rigid bodies) A and B. When B's orientation in A changes in such a way that there exists a unit vector $\hat{\boldsymbol{\lambda}}$ that is <u>fixed</u> in both A and B, then B is said to have a ***simple angular velocity*** in A given by equation (2), where θ is the angle from a vector fixed in A to a vector fixed in B, both of which are perpendicular to $\hat{\boldsymbol{\lambda}}$.[a] The sign ($\pm$) of $\dot{\theta}\hat{\boldsymbol{\lambda}}$ is determined from the right-hand rule. If increasing θ causes a right-hand rotation of B in A about $+\hat{\boldsymbol{\lambda}}$, the sign is positive, otherwise it is negative. Equation (2) is proved in Section 7.5.4.

$$\boxed{{}^{A}\vec{\omega}^{B} = \pm\,\dot{\theta}\,\hat{\boldsymbol{\lambda}}} \qquad (2)$$

When $\hat{\boldsymbol{\lambda}}$ is <u>fixed</u> in A and B

[a] A major obstacle in 3D kinematics is calculating angular velocity. Since angular velocity is complicated, most introductory textbooks specialize ***angular velocity*** to $\dot{\theta}\hat{\boldsymbol{\lambda}}$ and <u>**restrict**</u> $\hat{\boldsymbol{\lambda}}$ to be **fixed** in both A and B (i.e., planar 2D problems). For general 3D rotations [see equation (9.1) in Section 9.1], $\hat{\boldsymbol{\lambda}}$ is **not** fixed in A and B and ${}^{A}\vec{\omega}^{B} = \dot{\theta}\,\hat{\boldsymbol{\lambda}} + \sin(\theta)\,\frac{{}^{B}d\hat{\boldsymbol{\lambda}}}{dt} + [\cos(\theta)-1]\,\hat{\boldsymbol{\lambda}} \times \frac{{}^{B}d\hat{\boldsymbol{\lambda}}}{dt}$. When $\hat{\boldsymbol{\lambda}}$ is fixed in both A and B, $\frac{{}^{B}d\hat{\boldsymbol{\lambda}}}{dt} = \vec{0}$ and the general 3D expression simplifies to equation (2).

Simple angular velocity example - rolling wheel

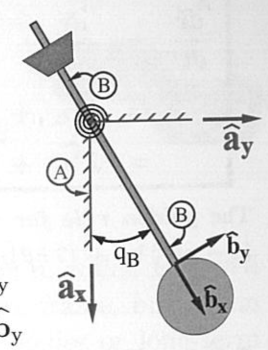


As shown in Section 7.3.8 and again to the right, wheel B has a simple angular velocity in airplane A of ${}^{A}\vec{\omega}^{B} = -\dot{\theta}\,\hat{\mathbf{b}}_z$.

Note: ${}^{A}\vec{\omega}^{B}$ is a ***simple angular velocity*** because $\hat{\mathbf{b}}_z$ is <u>fixed</u> in both A and B.

Note: there is no point of B that is continually fixed in A, i.e., B is <u>not</u> "***rotating about its center***" in A.

Example: Simple angular velocity ${}^{A}\vec{\omega}^{B}$ for a pendulum

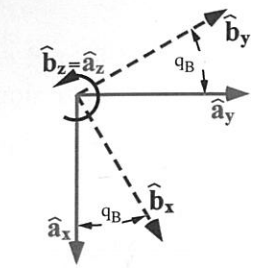


The figure to the right shows a rigid metronome B that swings in a reference frame A. Right-handed sets of orthogonal unit vectors $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ and $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ are fixed in B and A, respectively. The angle q_B is the angle from $\hat{\mathbf{a}}_x$ to $\hat{\mathbf{b}}_x$ with $+\hat{\mathbf{a}}_z$ sense.

1. Identify a unit vector $\hat{\boldsymbol{\lambda}}$ that is fixed in both A and B: $\hat{\boldsymbol{\lambda}} = \hat{\mathbf{b}}_z = \hat{\mathbf{a}}_z$
2. Identify a vector $\vec{\mathbf{a}}_\perp$ fixed in A and perpendicular to $\hat{\boldsymbol{\lambda}}$: $\vec{\mathbf{a}}_\perp = \hat{\mathbf{a}}_x$ or $\vec{\mathbf{a}}_\perp = \hat{\mathbf{a}}_y$
3. Identify a vector $\vec{\mathbf{b}}_\perp$ fixed in B and perpendicular to $\hat{\boldsymbol{\lambda}}$: $\vec{\mathbf{b}}_\perp = \hat{\mathbf{b}}_x$ or $\vec{\mathbf{b}}_\perp = \hat{\mathbf{b}}_y$
4. Identify the angle θ between $\vec{\mathbf{a}}_\perp$ and $\vec{\mathbf{b}}_\perp$ and calculate its time-derivative.
5. Use the right-hand rule to determine the sign of $\hat{\boldsymbol{\lambda}}$. In other words, point the four fingers of your right hand in the direction of $\vec{\mathbf{a}}_\perp$, and then curl them in the direction of $\vec{\mathbf{b}}_\perp$. If your thumb points in the direction of $\hat{\boldsymbol{\lambda}}$, the sign of $\hat{\boldsymbol{\lambda}}$ is positive, otherwise it is negative.

The chart shows various choices for $\vec{\mathbf{a}}_\perp$ and $\vec{\mathbf{b}}_\perp$ and corresponding expressions for θ (the angle between $\vec{\mathbf{a}}_\perp$ and $\vec{\mathbf{b}}_\perp$) and the sign of $\hat{\boldsymbol{\lambda}}$. Notice ${}^{A}\vec{\omega}^{B}$ ***does not*** depend on choice of $\vec{\mathbf{a}}_\perp$ and $\vec{\mathbf{b}}_\perp$.

$\vec{\mathbf{a}}_\perp$	$\vec{\mathbf{b}}_\perp$	θ	$\dot{\theta}$	$\pm\,\hat{\boldsymbol{\lambda}}$	${}^{A}\vec{\omega}^{B}$
$\hat{\mathbf{a}}_x$	$\hat{\mathbf{b}}_x$	q_B	$\dot{q}_B$	+	$\dot{q}_B\,\hat{\mathbf{b}}_z$
$\hat{\mathbf{a}}_x$	$\hat{\mathbf{b}}_y$	$\frac{\pi}{2} + q_B$	$\dot{q}_B$	+	$\dot{q}_B\,\hat{\mathbf{b}}_z$
$\hat{\mathbf{a}}_y$	$\hat{\mathbf{b}}_x$	$\frac{\pi}{2} - q_B$	$-\dot{q}_B$	-	$\dot{q}_B\,\hat{\mathbf{b}}_z$
$\hat{\mathbf{a}}_y$	$\hat{\mathbf{b}}_y$	q_B	$\dot{q}_B$	+	$\dot{q}_B\,\hat{\mathbf{b}}_z$



7.3.4 Angular velocity negative property

B's angular velocity in A is the negative of A's angular velocity in B.[a]

$$\boxed{{}^{A}\vec{\omega}^{B} = -{}^{B}\vec{\omega}^{A}} \qquad (3)$$

[a] The proof of equation (3) is in Section 7.5.6.

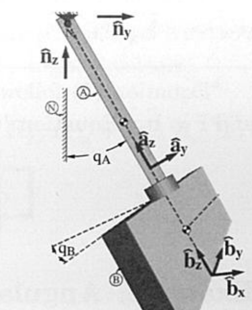
7.3.5 Angular velocity addition theorem

The ***angular velocity addition theorem*** relates the angular velocities of reference frames A, B, C, and D.

$$\boxed{{}^{A}\vec{\omega}^{D} = {}^{A}\vec{\omega}^{B} + {}^{B}\vec{\omega}^{C} + {}^{C}\vec{\omega}^{D}} \qquad (4)$$

Equation (4) is proved in Section 7.5.5.

7.3.6 Angular velocity example: Chaotic plate pendulum



The figure to the right shows a plate B connected by a revolute joint to a rod A so that B rotates freely about A's long axis. Rod A is connected to a fixed support N by another revolute joint. Note: The revolute joints' axes are perpendicular *not* parallel. Right-handed orthogonal unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ are fixed in A with $\hat{\mathbf{a}}_x$ parallel to the revolute joint connecting A to N and $\hat{\mathbf{a}}_z$ parallel to the axis of the revolute joint connecting B to A. To calculate ${}^{N}\vec{\omega}^{B}$, proceed as follows.[a]

- ${}^{N}\vec{\omega}^{A} \underset{(2)}{=} \dot{q}_A\,\hat{\mathbf{a}}_x$ A has a simple angular velocity in N since $\hat{\mathbf{a}}_x$ is **fixed** in both A and N.
- ${}^{A}\vec{\omega}^{B} \underset{(2)}{=} \dot{q}_B\,\hat{\mathbf{a}}_z$ B has a simple angular velocity in A since $\hat{\mathbf{a}}_z$ is **fixed** in both B and A.
- ${}^{N}\vec{\omega}^{B} \underset{(4)}{=} {}^{N}\vec{\omega}^{A} + {}^{A}\vec{\omega}^{B} = \dot{q}_A\,\hat{\mathbf{a}}_x + \dot{q}_B\,\hat{\mathbf{a}}_z$ Angular velocity addition theorem.

[a] Many textbooks define angular velocity as $\dot{\theta}\hat{\boldsymbol{\lambda}}$. This definition encounters serious obstacles in proving the angular velocity addition theorem of equation (4) and the golden rule for vector differentiation in equation (1). It is also deficient in calculations. For example, to start with ${}^{N}\vec{\omega}^{B} = \dot{\theta}\,\hat{\boldsymbol{\lambda}}$ and to arrive at ${}^{N}\vec{\omega}^{B} = \dot{q}_A\,\hat{\mathbf{a}}_x + \dot{q}_B\,\hat{\mathbf{a}}_z$ one has to discover a (non-physical) angle θ whose time-derivative is $\dot{\theta} = \sqrt{\dot{q}_A{}^2 + \dot{q}_B{}^2}$ and to somehow know that $\hat{\boldsymbol{\lambda}} = \frac{\dot{q}_A\,\hat{\mathbf{a}}_x + \dot{q}_B\,\hat{\mathbf{a}}_z}{\sqrt{\dot{q}_A{}^2+\dot{q}_B{}^2}}$.

7.3.7 Optional: New formula for angular velocity (Mitiguy, 1995)

It is difficult to **calculate** ${}^{A}\vec{\omega}^{B}$ (B's angular velocity in A) directly from its defining property in equation (1). A general way to calculate ${}^{A}\vec{\omega}^{B}$ is to use ***any non-coplanar*** vectors $\vec{\mathbf{p}}$, $\vec{\mathbf{q}}$, and $\vec{\mathbf{r}}$, and the time-derivatives in reference frames (or rigid vector bases) A and B, as[2]

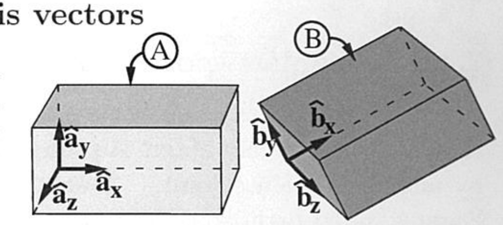
$$\boxed{\begin{aligned} {}^{A}\vec{\omega}^{B} &= \left[\left(\frac{{}^{A}d\vec{\mathbf{q}}}{dt} - \frac{{}^{B}d\vec{\mathbf{q}}}{dt}\right)\cdot\vec{\mathbf{r}} * \vec{\mathbf{p}} \;+\; \left(\frac{{}^{A}d\vec{\mathbf{r}}}{dt} - \frac{{}^{B}d\vec{\mathbf{r}}}{dt}\right)\cdot\vec{\mathbf{p}} * \vec{\mathbf{q}} \;+\; \left(\frac{{}^{A}d\vec{\mathbf{p}}}{dt} - \frac{{}^{B}d\vec{\mathbf{p}}}{dt}\right)\cdot\vec{\mathbf{q}} * \vec{\mathbf{r}} \right] / \left(\vec{\mathbf{p}} \times \vec{\mathbf{q}} \cdot \vec{\mathbf{r}}\right) \\ &= \left(\frac{{}^{A}d\vec{\mathbf{q}}}{dt} - \frac{{}^{B}d\vec{\mathbf{q}}}{dt}\right)\cdot\vec{\bar{r}} * \vec{\mathbf{p}} \;-\; \left(\frac{{}^{A}d\vec{\mathbf{p}}}{dt} - \frac{{}^{B}d\vec{\mathbf{p}}}{dt}\right)\cdot\vec{\bar{r}} * \vec{\mathbf{q}} \;+\; \left(\frac{{}^{A}d\vec{\mathbf{p}}}{dt} - \frac{{}^{B}d\vec{\mathbf{p}}}{dt}\right)\cdot\vec{\bar{q}} * \vec{\bar{r}} \end{aligned}} \qquad (5)$$

Equation (5) has features that deserve special attention.

[2] The proof of equation (5) is in Section 7.5.2. To our knowledge, the first appearance of equation (5) was communicated from Paul Mitiguy to Thomas Kane in 1995. If you find an earlier source, please let us know.

1. The second expression in equation (5) calculates ${}^{A}\vec{\omega}^{B}$ using only **two non-parallel** vectors $\vec{p}$ and $\vec{q}$ and their time-derivatives in A and B by defining $\vec{r} \triangleq \vec{p} \times \vec{q}$ and $\vec{\dot{r}} \triangleq \frac{\vec{p} \times \vec{q}}{(\vec{p} \times \vec{q})^2}$.
2. Equation (5) is **basis independent** as the vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$ are *any* non-coplanar vectors. Equation (5) does not involve matrices, unit vectors, bases, coordinate systems, angles or position vectors. For example, $\vec{p}$ can be a unit vector, position vector, velocity, acceleration, or angular momentum.
3. The denominator in equation (5) is non-zero as long as $\vec{p}$, $\vec{q}$, and $\vec{r}$ are non-coplanar. This has substantial computational advantages over [43, eqn. (1), pg. 68].
4. Equation (5) requires a minimal amount of information, namely two non-parallel vectors and their time-derivatives in A and B. This formula is more general than the basis-dependent formula for ${}^{A}\vec{\omega}^{B}$ first found in [35, eqn. 2.1, pg. 16] and later found in [56].
5. This formula simplifies when $\vec{r}$ is a unit vector **fixed** in both A and B.[3] This special case give rise to many authors' overly-simplistic definition of angular velocity, namely $\dot{\theta}\,\hat{\mathbf{r}}$. This simplistic definition does *not* lead to the very important property of angular velocity in equation (1), and its use is generally limited to *planar* (two-dimensional) kinematic analysis.

7.3.8 Optional: Angular velocity and orthogonal basis vectors



One way to calculate the angular velocity of a reference frame B in a reference frame A is to fix right-handed sets of orthogonal unit vectors $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ in B and perform the operations in equation (6).[a]

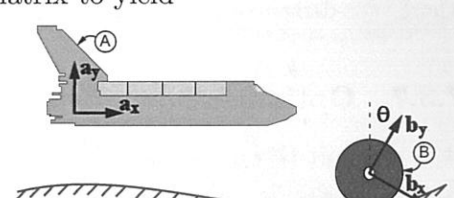
[a] Equation (6) follows directly from equation (5) by using $\vec{p} = \hat{b}_x$, $\vec{q} = \hat{b}_y$, and $\vec{r} = \hat{b}_z$. Equation (6) is also proved separately in Section 7.5.3.

$$\boxed{{}^{A}\vec{\omega}^{B} \;=\; \Big(\frac{{}^{A}d\hat{b}_y}{dt} \cdot \hat{b}_z\Big)\,\hat{b}_x \;+\; \Big(\frac{{}^{A}d\hat{b}_z}{dt} \cdot \hat{b}_x\Big)\,\hat{b}_y \;+\; \Big(\frac{{}^{A}d\hat{b}_x}{dt} \cdot \hat{b}_y\Big)\,\hat{b}_z} \tag{6}$$

Example: Angular velocity and orthogonal basis vectors

The following figure shows a rigid aircraft A in level horizontal flight above a hilly road and a rigid wheel B rolling to the right on the road. Right-handed orthogonal unit vectors $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ are fixed in A and B, respectively and ${}^{a}R^{b}$ is the rotation matrix relating $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ to $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$.

To calculate ${}^{A}\vec{\omega}^{B}$ with equation (6), use the definition of vector differentiation [equation (6.3)] for the vector derivatives in equation (6) in conjunction with the rotation matrix to yield



$$\frac{{}^{A}d\hat{b}_x}{dt} \underset{(6.3)}{=} -\sin(\theta)\,\dot{\theta}\,\hat{a}_x \;-\; \cos(\theta)\,\dot{\theta}\,\hat{a}_y \;=\; -\dot{\theta}\,\hat{b}_y$$

$$\frac{{}^{A}d\hat{b}_y}{dt} \underset{(6.3)}{=} \cos(\theta)\,\dot{\theta}\,\hat{a}_x \;+\; -\sin(\theta)\,\dot{\theta}\,\hat{a}_y \;=\; \dot{\theta}\,\hat{b}_x$$

$$\frac{{}^{A}d\hat{b}_z}{dt} \underset{(6.3)}{=} \vec{0}$$

${}^{a}R^{b}$	$\hat{b}_x$	$\hat{b}_y$	$\hat{b}_z$
$\hat{a}_x$	$\cos(\theta)$	$\sin(\theta)$	0
$\hat{a}_y$	$-\sin(\theta)$	$\cos(\theta)$	0
$\hat{a}_z$	0	0	1

Substituting these derivatives into the expression for ${}^{A}\vec{\omega}^{B}$ gives

$${}^{A}\vec{\omega}^{B} \;=\; (\dot{\theta}\,\hat{b}_x \cdot \hat{b}_z)\hat{b}_x \;+\; (\vec{0} \cdot \hat{b}_x)\hat{b}_y \;+\; (-\dot{\theta}\,\hat{b}_y \cdot \hat{b}_y)\hat{b}_z \;=\; \boxed{-\dot{\theta}\,\hat{b}_z}$$

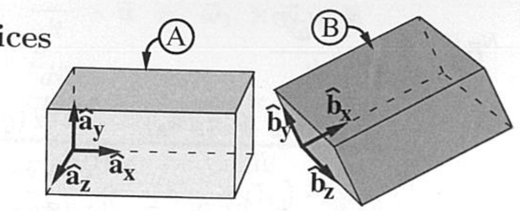
Before leaving this example, there are three concepts worth mentioning.

- ${}^{A}\vec{\omega}^{B}$ is a ***simple angular velocity*** because the vector $\hat{a}_z$ is **fixed** in both A and B. See Section 7.3.3 for a more efficient method for calculating simple angular velocity.

[3] A unit vector is fixed in a reference frame when its direction is invariant in that reference frame.

- Since B's center is not fixed in A, it is improper to say B ***"rotates about its center"*** in A. There is no point of B that continually remains fixed in A. See Homework 9.14 for ***instant centers***.
- In general, rotation can be studied separately from (and before) translation.

7.3.9 Optional: Angular velocity and rotation matrices



It directly follows from equation (6) that ${}^{A}\vec{\omega}^{B}$ can be calculated from the elements of the ${}^{a}R^{b}$ rotation matrix (relating two sets of right-handed orthogonal unit vectors $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$).

${}^{a}R^{b}$	$\hat{b}_x$	$\hat{b}_y$	$\hat{b}_z$
$\hat{a}_x$	R_{xx}	R_{xy}	R_{xz}
$\hat{a}_y$	R_{yx}	R_{yy}	R_{yz}
$\hat{a}_z$	R_{zx}	R_{zy}	R_{zz}

$$\boxed{\begin{aligned} {}^{A}\vec{\omega}^{B} \underset{(6)}{=}\; & (R_{xz}\,\dot{R}_{xy} + R_{yz}\,\dot{R}_{yy} + R_{zz}\,\dot{R}_{zy})\,\hat{b}_x \\ +\; & (R_{xx}\,\dot{R}_{xz} + R_{yx}\,\dot{R}_{yz} + R_{zx}\,\dot{R}_{zz})\,\hat{b}_y \\ +\; & (R_{xy}\,\dot{R}_{xx} + R_{yy}\,\dot{R}_{yx} + R_{zy}\,\dot{R}_{zx})\,\hat{b}_z \end{aligned}} \tag{7}$$

Because equation (7) contains time-derivatives of elements of the ${}^{a}R^{b}$ rotation matrix, it is clear that angular velocity is a measure of the time-rate of change of B's orientation in A.

Optional: Example: Angular velocity and rotation matrices

Referring to the previous example (Section 7.3.8), it is possible (albeit inefficient) to use equation (7) to calculate ${}^{A}\vec{\omega}^{B}$ as follows. See Section 7.3.3 for a more efficient method for calculating simple angular velocity.

$${}^{A}\vec{\omega}^{B} \underset{(7)}{=} (R_{xz}\dot{R}_{xy} + R_{yz}\dot{R}_{yy} + R_{zz}\dot{R}_{zy})\hat{b}_x + (R_{xx}\dot{R}_{xz} + R_{yx}\dot{R}_{yz} + R_{zx}\dot{R}_{zz})\hat{b}_y + (R_{xy}\dot{R}_{xx} + R_{yy}\dot{R}_{yx} + R_{zy}\dot{R}_{zx})\hat{b}_z$$
$$= (0+0+0)\,\hat{b}_x \;+\; (0+0+0)\,\hat{b}_y \;+\; \Big[-\sin^2(\theta)\,\dot{\theta} \;-\; \cos^2(\theta)\,\dot{\theta} \;+\; 0\Big]\,\hat{b}_z \;=\; \boxed{-\dot{\theta}\,\hat{b}_z}$$

7.4 Angular acceleration

Consider two reference frames (or rigid vector bases) A and B. B's ***angular acceleration*** in A is defined in equation (8).

$$\boxed{{}^{A}\vec{\alpha}^{B} \;\triangleq\; \frac{{}^{A}d\,{}^{A}\vec{\omega}^{B}}{dt}} \tag{8}$$

Applying the golden rule for vector differentiation in equation (1) to equation (8) yields an alternate (and sometimes computationally more efficient) expression for ${}^{A}\vec{\alpha}^{B}$.

$$\boxed{{}^{A}\vec{\alpha}^{B} \underset{(1,8)}{=} \frac{{}^{B}d\,{}^{A}\vec{\omega}^{B}}{dt}} \tag{9}$$

Note: *Points and particles do not have angular acceleration.*

7.4.1 Angular acceleration addition theorem

The ***angular acceleration addition theorem*** relates the angular accelerations of reference frames (or rigid vector bases) A, B and C as shown in equation (10).

$$\boxed{{}^{A}\vec{\alpha}^{C} = {}^{A}\vec{\alpha}^{B} + {}^{B}\vec{\alpha}^{C} + {}^{A}\vec{\omega}^{B} \times {}^{B}\vec{\omega}^{C}} \tag{10}$$

The proof is described in Homework 6.24.

Shown right: Although the angular velocity addition theorem is analogous to rotation matrix multiplication, the 3D angular acceleration addition theorem is more complicated.

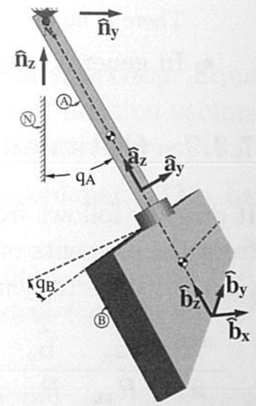
$$\boxed{\begin{aligned} {}^{a}R^{d} &= {}^{a}R^{b} \,*\, {}^{b}R^{c} \,*\, {}^{c}R^{d} \\ {}^{A}\vec{\omega}^{D} &= {}^{A}\vec{\omega}^{B} + {}^{B}\vec{\omega}^{C} + {}^{C}\vec{\omega}^{D} \\ {}^{A}\vec{\alpha}^{D} &= {}^{A}\vec{\alpha}^{B} + {}^{B}\vec{\alpha}^{C} + {}^{C}\vec{\alpha}^{D} + \underline{\text{other terms!}} \end{aligned}}$$

7.4.2 Angular acceleration negative property

B's angular acceleration in A is the negative of A's angular acceleration in B.

$$\boxed{{}^{A}\vec{\alpha}^{B} = -{}^{B}\vec{\alpha}^{A}} \tag{11}$$

7.4.3 Angular acceleration example: Chaotic plate pendulum



The figure to the right shows the system considered in Section 7.3.6. Using the definition of B's angular acceleration in N, ${}^N\vec{\alpha}^B$ is calculated as[a]

$$\begin{aligned}
{}^N\vec{\alpha}^B &\underset{(8)}{\triangleq} \frac{{}^N d\,{}^N\vec{\omega}^B}{dt} \\
&= \frac{{}^N d\,(\dot q_A\,\hat{a}_x + \dot q_B\,\hat{a}_z)}{dt} \underset{(1)}{=} \frac{{}^A d\,(\dot q_A\,\hat{a}_x + \dot q_B\,\hat{a}_z)}{dt} + {}^N\vec{\omega}^A \times (\dot q_A\,\hat{a}_x + \dot q_B\,\hat{a}_z) \\
&= \ddot q_A\,\hat{a}_x + \ddot q_B\,\hat{a}_z - \dot q_A\,\dot q_B\,\hat{a}_y
\end{aligned}$$

[a]Note: Even when $\dot q_A$ and $\dot q_B$ are **constant** (i.e., $\ddot q_A = 0$ and $\ddot q_B = 0$), ${}^N\vec{\alpha}^B \neq \vec{0}$.

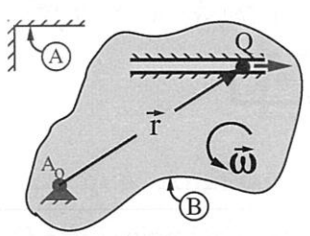
7.4.4 Optional: Second time-derivative of a vector in a reference frame

The 2^{nd} time-derivative of ***any*** vector $\vec{v}$ in a reference frame (or rigid vector basis) A can be calculated as

$$\boxed{\frac{{}^A d^2\vec{v}}{dt^2} = \frac{{}^B d^2\vec{v}}{dt^2} + {}^A\vec{\alpha}^B \times \vec{v} + {}^A\vec{\omega}^B \times ({}^A\vec{\omega}^B \times \vec{v}) + 2\,{}^A\vec{\omega}^B \times \frac{{}^B d\vec{v}}{dt}} \qquad (12)$$

The proof of equation (12) is in Section 7.5.7.

Example: Consider the position vector $\vec{r}$ from a point fixed in A to a point Q moving on a rigid body B.



Q's velocity and acceleration in A can be calculated ${}^A\vec{v}^Q = \frac{{}^A d\vec{r}}{dt}$ and ${}^A\vec{a}^Q = \frac{{}^A d^2\vec{r}}{dt^2}$, written in an efficient (and perhaps more familiar) way as

$$\begin{aligned}
{}^A\vec{v}^Q &\underset{(1)}{=} \underbrace{{}^B\vec{v}^Q}_{\textit{relative}} + \underbrace{{}^A\vec{\omega}^B \times \vec{r}}_{\textit{tangential}} \\
{}^A\vec{a}^Q &\underset{(12)}{=} \underbrace{{}^B\vec{a}^Q}_{\textit{relative}} + \underbrace{{}^A\vec{\alpha}^B \times \vec{r}}_{\textit{tangential}} + \underbrace{{}^A\vec{\omega}^B \times ({}^A\vec{\omega}^B \times \vec{r})}_{\textit{centripetal}} + \underbrace{2\,{}^A\vec{\omega}^B \times {}^B\vec{v}^Q}_{\textit{Coriolis}}
\end{aligned}$$

7.5 Optional: Angular velocity proofs

7.5.1 Proof of existence and uniqueness of the defining property of angular velocity

The proof of existence and uniqueness of the ***golden rule for vector differentiation*** in equation (1) begins with the ***vector dot-product derivative property*** presented in Section 6.6 and proved in Section 6.12, namely for arbitrary vectors $\vec{u}$ and $\vec{v}$ and arbitrary reference frames (or rigid bases) A or B,

$$\boxed{\frac{d(\vec{u}\cdot\vec{v})}{dt} = \frac{{}^A d\vec{u}}{dt}\cdot\vec{v} + \vec{u}\cdot\frac{{}^A d\vec{v}}{dt} = \frac{{}^B d\vec{u}}{dt}\cdot\vec{v} + \vec{u}\cdot\frac{{}^B d\vec{v}}{dt}} \qquad (13)$$

Substituting $\vec{v}$ for $\vec{u}$ in the previous equation is used to show $\frac{d\vec{v}}{dt}\cdot\vec{v}$ is independent of A or B:

$$\frac{d(\vec{v}\cdot\vec{v})}{dt} \underset{(13)}{=} 2\frac{{}^A d\vec{v}}{dt}\cdot\vec{v} \underset{(13)}{=} 2\frac{{}^B d\vec{v}}{dt}\cdot\vec{v} \quad\Rightarrow\quad \frac{{}^A d\vec{v}}{dt}\cdot\vec{v} = \frac{{}^B d\vec{v}}{dt}\cdot\vec{v} \quad\Rightarrow\quad \left(\frac{{}^A d\vec{v}}{dt} - \frac{{}^B d\vec{v}}{dt}\right)\cdot\vec{v} = 0$$

Since the difference $\frac{{}^A d\vec{v}}{dt} - \frac{{}^B d\vec{v}}{dt}$ is **perpendicular** to $\vec{v}$ and since the cross-product of an as-of-yet **unknown** vector $\vec{\omega}_1$ with $\vec{v}$ produces a vector perpendicular to $\vec{v}$, one can write the next equation below. One can then form the dot-product of this next equation with an arbitrary vector $\vec{u}$.

This process is repeated in the second line below – starting with $\vec{u}$, introducing another **unknown** vector $\vec{\omega}_2$, and then doing the dot-product with $\vec{v}$.

$$\begin{aligned}
\frac{{}^A d\vec{v}}{dt} - \frac{{}^B d\vec{v}}{dt} &= \vec{\omega}_1 \times \vec{v} &\Rightarrow&\quad \frac{{}^A d\vec{v}}{dt}\cdot\vec{u} - \frac{{}^B d\vec{v}}{dt}\cdot\vec{u} = \vec{\omega}_1 \times \vec{v}\cdot\vec{u} \\
\frac{{}^A d\vec{u}}{dt} - \frac{{}^B d\vec{u}}{dt} &= \vec{\omega}_2 \times \vec{u} &\Rightarrow&\quad +\ \frac{{}^A d\vec{u}}{dt}\cdot\vec{v} - \frac{{}^B d\vec{u}}{dt}\cdot\vec{v} = \vec{\omega}_2 \times \vec{u}\cdot\vec{v} \\
&&&\quad \frac{d(\vec{v}\cdot\vec{u})}{dt} - \frac{d(\vec{v}\cdot\vec{u})}{dt} = (\vec{\omega}_2 - \vec{\omega}_1)\cdot(\vec{u}\times\vec{v})
\end{aligned}$$

Adding the 1^{st} and 2^{nd} right-most equations in the previous set and using the vector-derivative property in equation (13) twice produces the last equation in the previous set (whose left-hand side simplifies to zero). Since both $\vec{u}$ and $\vec{v}$ are arbitrary vectors, $\vec{u}\times\vec{v}$ is neither $\vec{0}$ or perpendicular to the difference of the as-of-yet unknown vectors $\vec{\omega}_2 - \vec{\omega}_1$. Thus,

$$0 = (\vec{\omega}_2 - \vec{\omega}_1)\cdot(\vec{u}\times\vec{v}) \quad\Rightarrow\quad \vec{\omega}_2 - \vec{\omega}_1 = \vec{0}$$

Since $\vec{\omega}_2 = \vec{\omega}_1$, the proof of uniqueness for angular velocity is complete and the subscripts 1 and 2 are removed from the $\vec{\omega}$ symbol. Since $\vec{\omega}$ is associated with vector derivatives in reference frames (or rigid bases) A and B, $\vec{\omega}$ is denoted with superscripts A and B as ${}^A\vec{\omega}^B$.

7.5.2 Proof of the new formula for angular velocity

The proof of equation (5) begins by introducing three arbitrary non-coplanar vectors $\vec{p}$, $\vec{q}$, $\vec{r}$ (which form an non-orthgonal vector basis). Section 4.6 shows that **any** vector (including ${}^A\vec{\omega}^B$) can be written in terms of $\vec{p}$, $\vec{q}$, $\vec{r}$, and the scalars α, β, γ as

$${}^A\vec{\omega}^B = \alpha\,\vec{p} + \beta\,\vec{q} + \gamma\,\vec{r} \qquad (14)$$

$$\alpha = \frac{{}^A\vec{\omega}^B\cdot\vec{q}\times\vec{r}}{\vec{p}\cdot\vec{q}\times\vec{r}} \qquad \beta = \frac{{}^A\vec{\omega}^B\cdot\vec{r}\times\vec{p}}{\vec{q}\cdot\vec{r}\times\vec{p}} \qquad \gamma = \frac{{}^A\vec{\omega}^B\cdot\vec{p}\times\vec{q}}{\vec{r}\cdot\vec{p}\times\vec{q}}$$

Rewriting α, β, and γ by interchanging dot-products and cross-products gives

$$\alpha = \frac{{}^A\vec{\omega}^B\times\vec{q}\cdot\vec{r}}{\vec{p}\times\vec{q}\cdot\vec{r}} \qquad \beta = \frac{{}^A\vec{\omega}^B\times\vec{r}\cdot\vec{p}}{\vec{p}\times\vec{q}\cdot\vec{r}} \qquad \gamma = \frac{{}^A\vec{\omega}^B\times\vec{p}\cdot\vec{q}}{\vec{p}\times\vec{q}\cdot\vec{r}} \qquad (15)$$

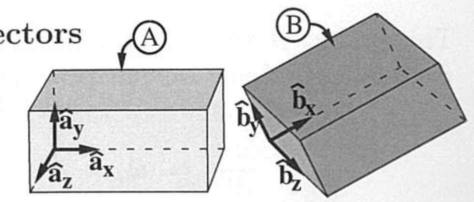
In view of equation (1), one may write

$$\begin{aligned}
\frac{{}^A d\vec{p}}{dt} &\underset{(1)}{=} \frac{{}^B d\vec{p}}{dt} + {}^A\vec{\omega}^B\times\vec{p} & & {}^A\vec{\omega}^B\times\vec{p} = \frac{{}^A d\vec{p}}{dt} - \frac{{}^B d\vec{p}}{dt} \qquad (16) \\
\frac{{}^A d\vec{q}}{dt} &\underset{(1)}{=} \frac{{}^B d\vec{q}}{dt} + {}^A\vec{\omega}^B\times\vec{q} &\Rightarrow\quad & {}^A\vec{\omega}^B\times\vec{q} = \frac{{}^A d\vec{q}}{dt} - \frac{{}^B d\vec{q}}{dt} \qquad (17) \\
\frac{{}^A d\vec{r}}{dt} &\underset{(1)}{=} \frac{{}^B d\vec{r}}{dt} + {}^A\vec{\omega}^B\times\vec{r} & & {}^A\vec{\omega}^B\times\vec{r} = \frac{{}^A d\vec{r}}{dt} - \frac{{}^B d\vec{r}}{dt} \qquad (18)
\end{aligned}$$

Substituting the right-hand sides of equations (16-18) into equation (15) produces the first expression in equation (5). The second expression in equation (5) results from the fact that for any vectors $\vec{p}$ and $\vec{r}$,

$$\frac{d(\vec{p}\cdot\vec{r})}{dt} = \frac{{}^A d\vec{p}}{dt}\cdot\vec{r} + \vec{p}\cdot\frac{{}^A d\vec{r}}{dt} = \frac{{}^B d\vec{p}}{dt}\cdot\vec{r} + \vec{p}\cdot\frac{{}^B d\vec{r}}{dt} \quad\text{or}\quad -\left(\frac{{}^A d\vec{p}}{dt} - \frac{{}^B d\vec{p}}{dt}\right)\cdot\vec{r} = \left(\frac{{}^A d\vec{r}}{dt} - \frac{{}^B d\vec{r}}{dt}\right)\cdot\vec{p}$$

7.5.3 Proof of angular velocity and orthogonal basis vectors



One way to calculate the angular velocity of a reference frame B in a reference frame A is to fix right-handed sets of orthogonal unit vectors $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ in B and perform the operations in equation (6).

A proof of equation (6) starts by applying equation (1) (the golden rule for vector differentiation) to each of $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ and noting $\frac{{}^{B}d\hat{\mathbf{b}}_i}{dt} = \vec{0}$ $(i = x, y, z)$. Secondly, the vector equations are dot-multiplied with $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$, $\hat{\mathbf{b}}_x$, respectively, (as shown below). Thirdly, equation (2.10) is used to rearrange the scalar triple products. Next, the relevant cross products are performed.

$$\frac{{}^{A}d\hat{\mathbf{b}}_x}{dt} \underset{(1)}{=} {}^{A}\vec{\omega}^{B} \times \hat{\mathbf{b}}_x \qquad \frac{{}^{A}d\hat{\mathbf{b}}_x}{dt} \cdot \hat{\mathbf{b}}_y = {}^{A}\vec{\omega}^{B} \times \hat{\mathbf{b}}_x \cdot \hat{\mathbf{b}}_y \underset{(2.10)}{=} {}^{A}\vec{\omega}^{B} \cdot \hat{\mathbf{b}}_x \times \hat{\mathbf{b}}_y = {}^{A}\vec{\omega}^{B} \cdot \hat{\mathbf{b}}_z$$

$$\frac{{}^{A}d\hat{\mathbf{b}}_y}{dt} \underset{(1)}{=} {}^{A}\vec{\omega}^{B} \times \hat{\mathbf{b}}_y \qquad \frac{{}^{A}d\hat{\mathbf{b}}_y}{dt} \cdot \hat{\mathbf{b}}_z = {}^{A}\vec{\omega}^{B} \times \hat{\mathbf{b}}_y \cdot \hat{\mathbf{b}}_z \underset{(2.10)}{=} {}^{A}\vec{\omega}^{B} \cdot \hat{\mathbf{b}}_y \times \hat{\mathbf{b}}_z = {}^{A}\vec{\omega}^{B} \cdot \hat{\mathbf{b}}_x$$

$$\frac{{}^{A}d\hat{\mathbf{b}}_z}{dt} \underset{(1)}{=} {}^{A}\vec{\omega}^{B} \times \hat{\mathbf{b}}_z \qquad \frac{{}^{A}d\hat{\mathbf{b}}_z}{dt} \cdot \hat{\mathbf{b}}_x = {}^{A}\vec{\omega}^{B} \times \hat{\mathbf{b}}_z \cdot \hat{\mathbf{b}}_x \underset{(2.10)}{=} {}^{A}\vec{\omega}^{B} \cdot \hat{\mathbf{b}}_z \times \hat{\mathbf{b}}_x = {}^{A}\vec{\omega}^{B} \cdot \hat{\mathbf{b}}_y$$

Equations (4.1) and (4.3) show ${}^{A}\vec{\omega}^{B}$ can be expressed in terms of $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ as given in the first equation below. Upon substitution of the previous expressions for ${}^{A}\vec{\omega}^{B} \cdot \hat{\mathbf{b}}_i$ $(i = x, y, z)$, one arrives at equation (6).

$$\begin{aligned} {}^{A}\vec{\omega}^{B} &= ({}^{A}\vec{\omega}^{B} \cdot \hat{\mathbf{b}}_x)\,\hat{\mathbf{b}}_x + ({}^{A}\vec{\omega}^{B} \cdot \hat{\mathbf{b}}_y)\,\hat{\mathbf{b}}_y + ({}^{A}\vec{\omega}^{B} \cdot \hat{\mathbf{b}}_z)\,\hat{\mathbf{b}}_z \\ &= (\frac{{}^{A}d\hat{\mathbf{b}}_y}{dt} \cdot \hat{\mathbf{b}}_z)\,\hat{\mathbf{b}}_x + (\frac{{}^{A}d\hat{\mathbf{b}}_z}{dt} \cdot \hat{\mathbf{b}}_x)\,\hat{\mathbf{b}}_y + (\frac{{}^{A}d\hat{\mathbf{b}}_x}{dt} \cdot \hat{\mathbf{b}}_y)\,\hat{\mathbf{b}}_z \end{aligned}$$

7.5.4 Proof of simple angular velocity

The following figure shows a rigid vector basis A consisting of right-handed, orthogonal, unit vectors $\hat{\boldsymbol{\alpha}}_1$, $\hat{\boldsymbol{\alpha}}_2$, $\hat{\boldsymbol{\lambda}}$; a rigid vector basis B consisting of right-handed, orthogonal unit vectors $\hat{\boldsymbol{\beta}}_1$, $\hat{\boldsymbol{\beta}}_2$, $\hat{\boldsymbol{\lambda}}$; and the rotation matrix relating these two sets of unit vectors. Note: $\hat{\boldsymbol{\lambda}}$ is fixed in both A and B.

To prove equation (2), start with equation (6) where $\hat{\boldsymbol{\beta}}_1$, $\hat{\boldsymbol{\beta}}_2$, $\hat{\boldsymbol{\lambda}}$ play the role of $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$, respectively.

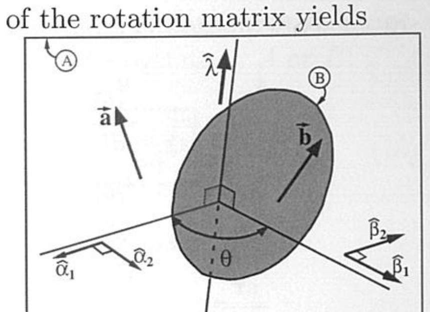
$${}^{A}\vec{\omega}^{B} \underset{(6)}{=} (\frac{{}^{A}d\hat{\boldsymbol{\beta}}_2}{dt} \cdot \hat{\boldsymbol{\lambda}})\,\hat{\boldsymbol{\beta}}_1 + (\frac{{}^{A}d\hat{\boldsymbol{\lambda}}}{dt} \cdot \hat{\boldsymbol{\beta}}_1)\,\hat{\boldsymbol{\beta}}_2 + (\frac{{}^{A}d\hat{\boldsymbol{\beta}}_1}{dt} \cdot \hat{\boldsymbol{\beta}}_2)\,\hat{\boldsymbol{\lambda}}$$

The definition of vector differentiation [equation (6.3)] and subsequent use of the rotation matrix yields

$$\frac{{}^{A}d\hat{\boldsymbol{\beta}}_1}{dt} \underset{(6.3)}{=} -\sin(\theta)\,\dot{\theta}\,\hat{\boldsymbol{\alpha}}_1 + \cos(\theta)\,\dot{\theta}\,\hat{\boldsymbol{\alpha}}_2 = \dot{\theta}\,\hat{\boldsymbol{\beta}}_2$$

$$\frac{{}^{A}d\hat{\boldsymbol{\beta}}_2}{dt} \underset{(6.3)}{=} -\cos(\theta)\,\dot{\theta}\,\hat{\boldsymbol{\alpha}}_1 + -\sin(\theta)\,\dot{\theta}\,\hat{\boldsymbol{\alpha}}_2 = -\dot{\theta}\,\hat{\boldsymbol{\beta}}_1$$

$$\frac{{}^{A}d\hat{\boldsymbol{\lambda}}}{dt} \underset{(6.3)}{=} \vec{0}$$



Substituting these derivatives into the expression for ${}^{A}\vec{\omega}^{B}$ gives

$${}^{A}\vec{\omega}^{B} = (-\dot{\theta}\,\hat{\boldsymbol{\beta}}_1 \cdot \hat{\boldsymbol{\lambda}})\,\hat{\boldsymbol{\beta}}_1 + (\vec{0} \cdot \hat{\boldsymbol{\beta}}_1)\,\hat{\boldsymbol{\beta}}_2 + (\dot{\theta}\,\hat{\boldsymbol{\beta}}_2 \cdot \hat{\boldsymbol{\beta}}_2)\,\hat{\boldsymbol{\lambda}} = \dot{\theta}\,\hat{\boldsymbol{\lambda}}$$

which shows ${}^{A}\vec{\omega}^{B} = \dot{\theta}\,\hat{\boldsymbol{\lambda}}$ and completes the proof of equation (2).

	$\hat{\boldsymbol{\alpha}}_1$	$\hat{\boldsymbol{\alpha}}_2$	$\hat{\boldsymbol{\lambda}}$
$\hat{\boldsymbol{\beta}}_1$	$\cos(\theta)$	$\sin(\theta)$	0
$\hat{\boldsymbol{\beta}}_2$	$-\sin(\theta)$	$\cos(\theta)$	0
$\hat{\boldsymbol{\lambda}}$	0	0	1

7.5.5 Proof of angular velocity addition theorem

The proof of equation (4) begins by introducing an ***arbitary*** vector $\vec{v}$ that is a function of time t in reference frames (or rigid vector bases) A, B, C, and D and using equation (1) to write each of the four relationships below.

$$\frac{{}^{A}d\vec{v}}{dt} \underset{(1)}{=} \frac{{}^{B}d\vec{v}}{dt} + {}^{A}\vec{\omega}^{B} \times \vec{v}$$

$$\frac{{}^{B}d\vec{v}}{dt} \underset{(1)}{=} \frac{{}^{C}d\vec{v}}{dt} + {}^{B}\vec{\omega}^{C} \times \vec{v}$$

$$\frac{{}^{C}d\vec{v}}{dt} \underset{(1)}{=} \frac{{}^{D}d\vec{v}}{dt} + {}^{C}\vec{\omega}^{D} \times \vec{v}$$

$$\frac{{}^{A}d\vec{v}}{dt} \underset{(1)}{=} \frac{{}^{D}d\vec{v}}{dt} + {}^{A}\vec{\omega}^{D} \times \vec{v}$$

These four relationships are then manipulated by: substituting the right-hand side of the last equation into the left-hand side of the first equation; substituting the right-hand side of the third equation into the right-hand side of the second equation and subsequently substituting into the right-hand side of the first equation. This gives

$$\begin{aligned} {}^{A}\vec{\omega}^{D} \times \vec{v} &= {}^{A}\vec{\omega}^{B} \times \vec{v} + {}^{B}\vec{\omega}^{C} \times \vec{v} + {}^{C}\vec{\omega}^{D} \times \vec{v} \\ &= ({}^{A}\vec{\omega}^{B} + {}^{B}\vec{\omega}^{C} + {}^{C}\vec{\omega}^{D}) \times \vec{v} \end{aligned}$$

which can only be satisfied for every vector $\vec{v}$ when

$${}^{A}\vec{\omega}^{D} = {}^{A}\vec{\omega}^{B} + {}^{B}\vec{\omega}^{C} + {}^{C}\vec{\omega}^{D}$$

7.5.6 Proof of angular velocity negative property

To prove equation (3), use equation (4) (the addition theorem) to write ${}^{A}\vec{\omega}^{A} = {}^{A}\vec{\omega}^{B} + {}^{B}\vec{\omega}^{A}$. Since ${}^{A}\vec{\omega}^{A} = \vec{0}$ (see below), ${}^{A}\vec{\omega}^{B} = -{}^{B}\vec{\omega}^{A}$ which completes the proof of equation (3).

To show ${}^{A}\vec{\omega}^{A} = \vec{0}$, note that equation (1) (the golden rule for vector differentiation) applied to ${}^{A}\vec{\omega}^{A}$ is

$$\frac{{}^{A}d\vec{v}}{dt} \underset{(1)}{=} \frac{{}^{A}d\vec{v}}{dt} + {}^{A}\vec{\omega}^{A} \times \vec{v} \quad \Rightarrow \quad \vec{0} = {}^{A}\vec{\omega}^{A} \times \vec{v}$$

The only way ${}^{A}\vec{\omega}^{A} \times \vec{v} = \vec{0}$ for *any* vector $\vec{v}$ is for ${}^{A}\vec{\omega}^{A} = \vec{0}$.

7.5.7 Proof of 2^{nd} time-derivative of any vector

The proof of equation (12) follows by differentiating equation (1) in A to give

$$\begin{aligned} \frac{{}^{A}d}{dt}\left(\frac{{}^{A}d\vec{v}}{dt}\right) &\underset{(1)}{=} \frac{{}^{A}d}{dt}\left(\frac{{}^{B}d\vec{v}}{dt} + {}^{A}\vec{\omega}^{B} \times \vec{v}\right) \\ &= \frac{{}^{A}d}{dt}\left(\frac{{}^{B}d\vec{v}}{dt}\right) + \frac{{}^{A}d\,{}^{A}\vec{\omega}^{B}}{dt} \times \vec{v} + {}^{A}\vec{\omega}^{B} \times \frac{{}^{A}d\vec{v}}{dt} \\ &\underset{(18)}{=} \frac{{}^{B}d}{dt}\left(\frac{{}^{B}d\vec{v}}{dt}\right) + {}^{A}\vec{\omega}^{B} \times \frac{{}^{B}d\vec{v}}{dt} + {}^{A}\vec{\alpha}^{B} \times \vec{v} + {}^{A}\vec{\omega}^{B} \times (\frac{{}^{B}d\vec{v}}{dt} + {}^{A}\vec{\omega}^{B} \times \vec{v}) \\ &= \frac{{}^{B}d^2\vec{v}}{dt^2} + {}^{A}\vec{\alpha}^{B} \times \vec{v} + {}^{A}\vec{\omega}^{B} \times \left({}^{A}\vec{\omega}^{B} \times \vec{v}\right) + 2\,{}^{A}\vec{\omega}^{B} \times \frac{{}^{B}d\vec{v}}{dt} \end{aligned}$$