Chapter 6

Vector differentiation and integration

Summary (see examples in Hw 5 and 6)

Many engineering analyses involve rates of change of vectors. For example, motion studies involve velocity (time-rate of change of position) and geometry involves curvature (spatial-rate of change of position). This chapter presents concepts and a precise definition for the ***derivative of a vector*** in a reference frame.

Note: A ***reference frame*** is simply a ***rigid object***. Reference frames and rigid bases are discussed in Section 7.2. For computationally efficiency, consider the ***golden rule for vector differentiation*** in Section 7.3.

6.1 Differentation concepts: Changes in magnitude and direction

In **scalar** calculus, $\frac{df}{dt}$ (the ordinary derivative of a scalar function f with respect to the scalar variable t) is defined as shown to the right.

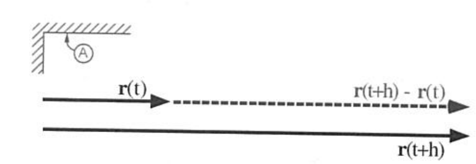
$$\frac{df}{dt} \triangleq \lim_{h \to 0} \frac{f(t+h) \; - \; f(t)}{h}$$

In **vector** calculus, $\frac{^A d\vec{\mathbf{v}}}{dt}$ [the ordinary derivative in ***reference frame*** (or ***rigid vector basis***) A of a vector $\vec{\mathbf{v}}$ with respect to the scalar variable t] is equal to the expression shown to the right where $\vec{\mathbf{v}}(t+h)|_A$ and $\vec{\mathbf{v}}(t)|_A$ denote $\vec{\mathbf{v}}$ evaluated in A at $t+h$ and t, respectively.

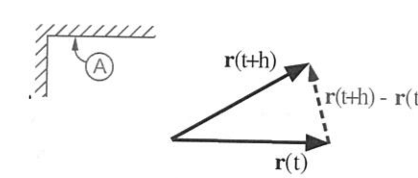
$$\frac{^A d\vec{\mathbf{v}}}{dt} \triangleq \lim_{h \to 0} \frac{\vec{\mathbf{v}}(t+h)|_A \; - \; \vec{\mathbf{v}}(t)|_A}{h}$$

Differentiating a **vector** is more complicated than differentiating a scalar because a vector's magnitude can change, its direction in reference frame (or rigid basis) A can change, or both can change.

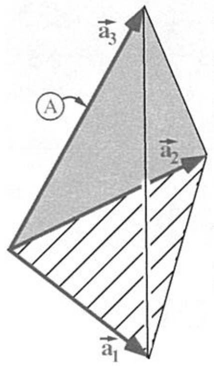
For example, the figure to the right shows a vector $\vec{\mathbf{r}}$ whose **magnitude changes** but whose direction in reference frame A remains constant . The vector $\vec{\mathbf{r}}(t+h) - \vec{\mathbf{r}}(t)$ measures the change in reference frame A of the vector $\vec{\mathbf{r}}$ from time t to time $t+h$. In the limit as $h \to 0$, the direction of $\frac{^A d\vec{\mathbf{r}}}{dt}$ is parallel to $\vec{\mathbf{r}}$.



The second example shows a vector $\vec{\mathbf{r}}$ whose **magnitude is constant** but whose direction in reference frame A changes. The vector $\vec{\mathbf{r}}(t+h) - \vec{\mathbf{r}}(t)$ measures the change in reference frame A of the vector $\vec{\mathbf{r}}$ from time t to time $t+h$. In the limit as $h \to 0$, $\frac{^A d\vec{\mathbf{r}}}{dt}$ is perpendicular to $\vec{\mathbf{r}}(t)$.



6.2 Expressing a vector in terms of vectors fixed in a reference frame



When three noncoplanar (but not necessarily orthogonal or unit) vectors $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ are *fixed* in a ***reference frame*** (or ***rigid vector basis***) A,[a] one can show (see Section 4.6) there exist three unique scalar functions v_1, v_2, v_3 such that *any* vector $\vec{v}$ can be *expressed* as

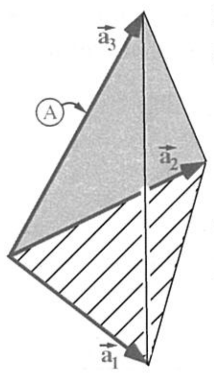
$$\vec{v} = v_1\,\vec{a}_1 + v_2\,\vec{a}_2 + v_3\,\vec{a}_3 \qquad (1)$$

When one or more of v_1, v_2, v_3 are a function of the scalar variable t, $\vec{v}$ is called a ***vector function*** of t in A and one may define the vector $\vec{v}$ ***evaluated in*** A ***at*** $t=\bar{t}$ as

$$\vec{v}(\bar{t})|_A \triangleq v_1(\bar{t})\,\vec{a}_1 + v_2(\bar{t})\,\vec{a}_2 + v_3(\bar{t})\,\vec{a}_3$$

[a] A ***reference frame*** (or ***rigid vector basis***) A can be constructed by as few as three non-collinear points P_1, P_2, P_3 whose distance from each other are constant. Reference frames and rigid bases are discussed in Section 7.2. Three noncoplanar vectors that are *inherently fixed* in A are: $\vec{a}_1$ from P_1 to P_2; $\vec{a}_2$ from P_1 to P_3; and $\vec{a}_3 = \vec{a}_1 \times \vec{a}_2$.

6.3 Partial and ordinary derivatives of a vector in a reference frame



Referring to Section 6.2, when v_1, v_2, v_3 are functions of the scalar variables s and t, one may define the ***partial derivative in*** A ***of*** $\vec{v}$ ***with respect to*** t as either

$$\boxed{\frac{{}^A\partial\vec{v}}{\partial t} \triangleq \frac{\partial v_1}{\partial t}\,\vec{a}_1 + \frac{\partial v_2}{\partial t}\,\vec{a}_2 + \frac{\partial v_3}{\partial t}\,\vec{a}_3} \quad \text{or} \quad \boxed{\frac{{}^A\partial\vec{v}}{\partial t} \underset{(11)}{\triangleq} \lim_{h\to 0} \frac{\vec{v}(s,t+h)|_A - \vec{v}(s,t)|_A}{h}} \qquad (2)$$

When v_1, v_2, v_3 are functions of a *single* scalar variable t, the ***ordinary derivative in*** A ***of*** $\vec{v}$ ***with respect to*** t is defined as either

$$\boxed{\frac{{}^Ad\vec{v}}{dt} \underset{(2)}{\triangleq} \frac{dv_1}{dt}\,\vec{a}_1 + \frac{dv_2}{dt}\,\vec{a}_2 + \frac{dv_3}{dt}\,\vec{a}_3} \quad \text{or} \quad \boxed{\frac{{}^Ad\vec{v}}{dt} \underset{(2)}{\triangleq} \lim_{h\to 0} \frac{\vec{v}(t+h)|_A - \vec{v}(t)|_A}{h}} \qquad (3)$$

6.4 Constant vectors (vectors fixed in a reference frame)

Referring to Section 6.3, when each of v_1, v_2, v_3 are *constant*, $\vec{v}$ is said to be a ***constant vector in*** A [or equivalently a ***vector fixed in*** A]. When $\vec{v}$ is constant in A,[1]

- $\vec{v}$ has a constant magnitude, i.e., $|\vec{v}| = C$ where C is a constant
- $\vec{v}$ has a constant direction in A, i.e., $\vec{v}\cdot\vec{a}_i = C_i$ where C_i is a constant and $\vec{a}_i$ is *any* vector fixed in A
- $\frac{{}^Ad\vec{v}}{dt} = \vec{0}$ [proved by inspection of equation (3)]

  Note: Certain analyses (e.g., ***conservation of linear momentum*** or ***conservation of angular momentum***) lead to expressions like $\frac{{}^Ad\vec{v}}{dt} = \vec{0}$. Information about $\vec{v}$ is determined by setting $|\vec{v}| = C$ or $\vec{v}\cdot\vec{a}_i = C_i$.

6.5 Vectors with constant magnitude

Since a vector's $\vec{v}$'s magnitude is a scalar, the change in $|\vec{v}|$ is *independent of reference frame or basis*. If $\vec{v}$ has ***constant magnitude***, $\frac{{}^Fd\vec{v}}{dt}$, the derivative of $\vec{v}$ in *any* reference frame (or rigid basis) F, is perpendicular to $\vec{v}$.

$$\boxed{\vec{v}\cdot\frac{{}^Fd\vec{v}}{dt} = 0} \qquad (4)$$

If $\vec{v}$'s magnitude is constant

Note: This is shown in equation (4) and verifies the second conceptual example in Section 6.1.

Proof of equation (4): $\vec{v}\cdot\vec{v}$ = constant, hence $\frac{d(\vec{v}\cdot\vec{v})}{dt} = 0$. Section 6.6 gives $\frac{d(\vec{v}\cdot\vec{v})}{dt} = 2\,\vec{v}\cdot\frac{{}^Fd\vec{v}}{dt}$.

[1] It does not make sense to state that a "vector is constant or fixed" without specifying a reference frame (or rigid basis).

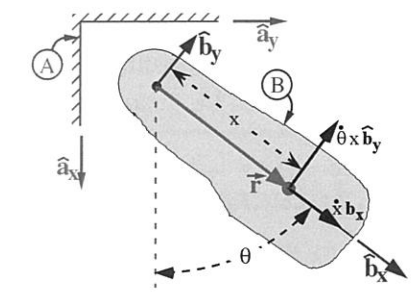
6.6 Properties of derivatives of vectors

The following are derivative properties for arbitrary vectors $\vec{u}$, $\vec{v}$, $\vec{w}$, an arbitrary dependent scalar variable s, an arbitrary independent variable t, and an arbitrary reference frame (or rigid basis) A.

Properties of ordinary or partial derivatives.	
$\frac{d(\vec{u}\cdot\vec{v})}{dt} = \frac{{}^Ad\vec{u}}{dt}\cdot\vec{v} + \vec{u}\cdot\frac{{}^Ad\vec{v}}{dt}$	$\frac{{}^Ad(s\,\vec{u})}{dt} = \frac{ds}{dt}\,\vec{u} + s\,\frac{{}^Ad\vec{u}}{dt}$
$\frac{{}^Ad(\vec{u}\times\vec{v})}{dt} = \frac{{}^Ad\vec{u}}{dt}\times\vec{v} + \vec{u}\times\frac{{}^Ad\vec{v}}{dt}$	$\frac{{}^Ad(\vec{u}+\vec{v}+\vec{w})}{dt} = \frac{{}^Ad\vec{u}}{dt} + \frac{{}^Ad\vec{v}}{dt} + \frac{{}^Ad\vec{w}}{dt}$
$\frac{{}^Ad(\vec{u}*\vec{v})}{dt} = \frac{{}^Ad\vec{u}}{dt}*\vec{v} + \vec{u}*\frac{{}^Ad\vec{v}}{dt}$	$\frac{d(\vec{u}\times\vec{v}\cdot\vec{w})}{dt} = \frac{{}^Ad\vec{u}}{dt}\times\vec{v}\cdot\vec{w} + \vec{u}\times\frac{{}^Ad\vec{v}}{dt}\cdot\vec{w} + \vec{u}\times\vec{v}\cdot\frac{{}^Ad\vec{w}}{dt}$

Section 6.12 proves the first equation i.e., the ***vector dot-product derivative property***.

6.7 Example: Derivatives of a vector



The derivative of a ***vector*** is substantially different than the derivative of a ***scalar*** because a vector derivative involves a change in direction whereas a scalar derivative does not.

To demonstrate how the derivative of a vector can involve a reference frame, consider a rigid body B that rotates in a plane A. Right-handed sets of orthogonal unit vectors $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ are fixed in A and B, respectively, with $\hat{a}_z = \hat{b}_z$ normal to the plane.

The orientation of $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ is determined by first setting $\hat{b}_i = \hat{a}_i$ ($i = x, y, z$) and then subjecting B to a right-handed rotation in A characterized by $\theta\,\hat{a}_z$ (θ is a variable that depends on time t).

${}^bR^a$	$\hat{a}_x$	$\hat{a}_y$	$\hat{a}_z$
$\hat{b}_x$	$\cos(\theta)$	$\sin(\theta)$	0
$\hat{b}_y$	$-\sin(\theta)$	$\cos(\theta)$	0
$\hat{b}_z$	0	0	1

(5)

Shown below is the calculation of the derivative of the vector $\vec{r} = x\,\hat{b}_x$ (x is a time-dependent variable) in both B and A. Since $\frac{{}^Bd\vec{r}}{dt} \neq \frac{{}^Ad\vec{r}}{dt}$, it is clear that *reference frames make a difference!* [2]

Time-derivative of $\vec{r}$ in B	Time-derivative of $\vec{r}$ in A
$\vec{r} = x\,\hat{b}_x$	$\vec{r} \underset{(5)}{=} x\,[\cos(\theta)\,\hat{a}_x + \sin(\theta)\,\hat{a}_y]$
$\frac{{}^Bd\vec{r}}{dt} \underset{(3)}{=} \dot{x}\,\hat{b}_x$	$\frac{{}^Ad\vec{r}}{dt} \underset{(3)}{=} \dot{x}\,[\cos(\theta)\,\hat{a}_x + \sin(\theta)\,\hat{a}_y] + x\,\dot{\theta}\,[-\sin(\theta)\,\hat{a}_x + \cos(\theta)\,\hat{a}_y]$
	$\underset{(5)}{=} \dot{x}\,\hat{b}_x + x\,\dot{\theta}\,\hat{b}_y$

In view of the fact that $\frac{{}^Ad\vec{r}}{dt}$ differs from $\frac{{}^Bd\vec{r}}{dt}$ by only one term (which is perpendicular to $\vec{r}$), it is natural to wonder how to relate them. The next chapter gives a *very* important relationship in equation (7.1) for derivatives in different reference frames called the ***golden rule for vector differentiation***, which for this example (worked out again in Section 7.3.1) is

$$\boxed{\frac{{}^Ad\vec{r}}{dt} = \frac{{}^Bd\vec{r}}{dt} + {}^A\vec{\omega}^B \times \vec{r}}$$

[2] Since $\vec{r}$ is a *vector*, its time-derivative describes its change in *magnitude* and *direction*. Since $\vec{r}$'s magnitude is a scalar, changes in magnitude can be analyzed with scalar calculus, e.g., the time-derivative of x is simply $\dot{x}$. However, to determine how $\vec{r}$'s direction changes, we must ask "with respect to what". For example, $\vec{r}$'s direction does not change in B since $\vec{r}$ is always in the $\hat{b}_x$ direction and $\hat{b}_x$ is *fixed* on B. Conversely, as B rotates in A, $\vec{r}$'s direction changes in A. The faster B spins in A, the faster $\vec{r}$'s direction changes in A. This may be demonstrated by spinning in a room while elongating a bike pump.

6.8 Differential of a vector

Referring to Section 6.2, when a vector $\vec{v}$ is regarded as a function of n independent scalar variables $t_1, \ldots, t_n$ in a reference frame (or rigid basis) A, one may define a quantity ${}^A d\vec{v}$ called the ***differential in** A **of** $\vec{v}$* in terms of $dt_1, \ldots, dt_n$ (***differentials of the independent variables*** $t_1, \ldots, t_n$). These "independent differentials" are defined to be arbitrary (usually small) quantities that have the same dimension of $t_1, \ldots, t_n$. With $dt_1, \ldots, dt_n$ in hand, ${}^A d\vec{v}$ is defined as either[3]

$$\boxed{{}^A d\vec{v} \triangleq dv_1\,\vec{a}_1 + dv_2\,\vec{a}_2 + dv_3\,\vec{a}_3} \quad \text{or} \quad \boxed{{}^A d\vec{v} \triangleq \frac{{}^A\partial\vec{v}}{\partial t_1}dt_1 + \frac{{}^A\partial\vec{v}}{\partial t_2}dt_2 + \ldots \frac{{}^A\partial\vec{v}}{\partial t_n}dt_n} \qquad (6)$$

When $\vec{v}$ is regarded as a function of a *single* scalar variable t in A, the right-most equation (6) reduces to the equation shown to the right. Subsequently dividing both sides by dt gives rise to the *ratio* of ${}^A d\vec{v}$ to dt.

$${}^A d\vec{v} \underset{(6)}{=} \frac{{}^A\partial\vec{v}}{\partial t}dt$$

Hence, although the symbol $\frac{{}^A d\vec{v}}{dt}$ can always be regarded as a ***ratio of differentials***, it can sometimes be an ***ordinary derivative*** in the sense of equation (3).

$$\frac{{}^A d\vec{v}}{dt} = \frac{{}^A\partial\vec{v}}{\partial t}$$

6.9 Integral of a vector

Referring to Section 6.8, when a vector $\vec{v}$ is regarded as a function of the scalar variable t in a ***reference frame*** (or ***rigid vector basis***) A, one may define the ***integral in** A **of** $\vec{v}$* as

$$\boxed{{}^A\!\!\int \vec{v}\,dt \triangleq \left(\int v_1\,dt\right)\vec{a}_1 + \left(\int v_2\,dt\right)\vec{a}_2 + \left(\int v_3\,dt\right)\vec{a}_3} \qquad (7)$$

For example, substituting the left-most expression of ${}^A d\vec{v}$ in equation (6) for $\vec{v}$ in equation (7) results in a definition for the ***integral in** A **of the differential of** $\vec{v}$*.[4]

$$\begin{aligned} {}^A\!\!\int {}^A d\vec{v} &\underset{(6,7)}{=} \left(\int dv_1\right)\vec{a}_1 + \left(\int dv_2\right)\vec{a}_2 + \left(\int dv_3\right)\vec{a}_3 \\ &= (v_1 + c_1)\,\vec{a}_1 + (v_2 + c_2)\,\vec{a}_2 + (v_3 + c_3)\,\vec{a}_3 \\ &= \vec{v} + \vec{c} \qquad \text{where } \vec{c} \text{ is a constant vector in } A \text{ (i.e., } \vec{c} \text{ is fixed in } A) \end{aligned} \qquad (8)$$

6.10 Optional**: Limit of a vector in a reference frame

Referring to Section 6.2, when $\vec{v}$ is a ***vector function*** in reference frame (or rigid basis) A of scalar variables s and t, the ***vector limit in** A **of** $\vec{v}$* as $t \to \bar{t}$ is defined as

$$\boxed{\lim_{t\to\bar{t}} \vec{v}(s,t)|_A \triangleq \left[\lim_{t\to\bar{t}} v_1(s,t)\right]\vec{a}_1 + \left[\lim_{t\to\bar{t}} v_2(s,t)\right]\vec{a}_2 + \left[\lim_{t\to\bar{t}} v_3(s,t)\right]\vec{a}_3} \qquad (9)$$

[3] The equivalence of the definitions of ${}^A d\vec{v}$ in equation (6) is shown by substituting $\frac{{}^A\partial\vec{v}}{\partial t_i} \underset{(2)}{\triangleq} \frac{\partial v_1}{\partial t_i}\vec{a}_1 + \frac{\partial v_2}{\partial t_i}\vec{a}_2 + \frac{\partial v_3}{\partial t_i}\vec{a}_3$ $(i = 1, 2, \ldots, n)$, into the right-most expression for ${}^A d\vec{v}$ in equation (6) and factoring on $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$. Next, the coefficient of $\vec{a}_1$ is seen as the differential of v_1, i.e., $dv_1 \underset{(1.26)}{\triangleq} \frac{\partial f}{\partial t_1}dt_1 + \frac{\partial f}{\partial t_2}dt_2 + \ldots \frac{\partial f}{\partial t_n}dt_n$. Similarly for the coefficients of $\vec{a}_2$, $\vec{a}_3$.

[4] Section 10.7 shows the utility of equation (8) for integrating acceleration to find velocity and position.

To connect vector limits with vector differentiation, apply the limit definition in equation (9) as shown to the right.

Next, use the definition in equation (1.25) for the ***partial derivative*** of the scalar v_i (i=1, 2, 3) with respect to t (as shown below).

$$\begin{aligned} \lim_{h\to 0} \frac{\vec{v}(s,t+h)|_A - \vec{v}(s,t)|_A}{h} \underset{(9)}{\triangleq} & \left[\lim_{h\to 0} \frac{v_1(s,t+h) - v_1(s,t)}{h}\right]\vec{a}_1 \\ + & \left[\lim_{h\to 0} \frac{v_2(s,t+h) - v_2(s,t)}{h}\right]\vec{a}_2 \\ + & \left[\lim_{h\to 0} \frac{v_3(s,t+h) - v_3(s,t)}{h}\right]\vec{a}_3 \end{aligned} \qquad (10)$$

Lastly, using the definition in equation (2) proves how vector limits are related to vector differentiation.

$$\lim_{h\to 0} \frac{\vec{v}(s,t+h)|_A - \vec{v}(s,t)|_A}{h} \underset{(10,\,1.25)}{=} \frac{\partial v_1}{\partial t}\vec{a}_1 + \frac{\partial v_2}{\partial t}\vec{a}_2 + \frac{\partial v_3}{\partial t}\vec{a}_3 \underset{(2)}{=} \frac{{}^A\partial\vec{v}}{\partial t} \qquad (11)$$

6.11 Optional**: Differentiation with respect to a vector and gradients

Sometimes a **scalar function** such as temperature depends on a **vector** such as a position vector.

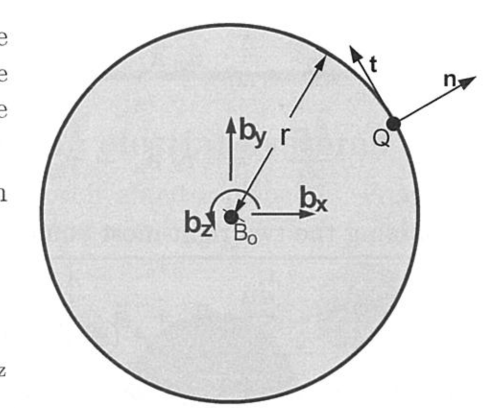
If a scalar F depends on a vector $\vec{v}$, it is useful to define the vector denoted $\vec{\nabla}_{\vec{v}}F$ in equation (12) where $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ are *any* orthogonal unit vectors and $v_i \triangleq \vec{v}\cdot\hat{a}_i$ $(i = x, y, z)$.

$$\boxed{\vec{\nabla}_{\vec{v}}F \triangleq \frac{\partial F}{\partial v_x}\hat{a}_x + \frac{\partial F}{\partial v_y}\hat{a}_y + \frac{\partial F}{\partial v_z}\hat{a}_z} \qquad (12)$$

The quantity $\vec{\nabla}_{\vec{v}}F$ is called ***differentiation of** F **with respect to** $\vec{v}$* and is invariant with respect to the choice of basis vectors $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$. When $\vec{\nabla}_{\vec{v}}F$ produces a force vector, the scalar function F is called a ***force function***. When $\vec{v}$ is a ***position vector***, $\vec{\nabla}_{\vec{v}}F$ is called a ***spatial gradient*** (and is frequently denoted without the subscript, i.e., $\vec{\nabla}F$). If F is a continuous function that describes the ***surface of an object***, $\vec{\nabla}_{\vec{v}}F$ is normal to the surface of the object.

Example of differential geometry: Normal and tangent to a circle

A ***circle*** can be defined as the locus of points in a plane that are a distance r (called the ***circle's radius***) from a point B_o (called the ***circle's center***). For example, the figure to the right shows a circle of radius r that is centered at point B_o.



The position of a point Q on the circle's periphery from point B_o can be expressed in terms of the scalars x and y as

$$\vec{r}^{\,Q/B_o} = x\,\hat{b}_x + y\,\hat{b}_y$$

where $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ are right-handed, orthogonal, unit vectors with $\hat{b}_z$ perpendicular to the plane of the circle.

A mathematical definition of a circle is $|\vec{r}^{\,Q/B_o}| = r$ which results in the scalar relationship F to the right between x, y, and r.

$$F = x^2 + y^2 - r^2 = 0$$

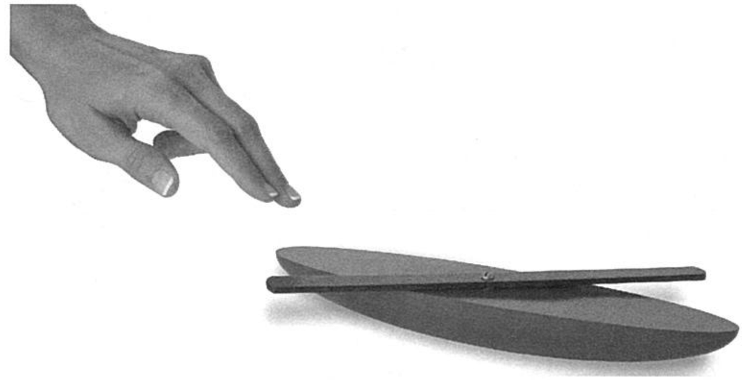
When a scalar function F describes the boundary of an object, the spatial gradient $\vec{\nabla}F$ is normal to the boundary. With $\vec{r}^{\,Q/B_o} = x\,\hat{b}_x + y\,\hat{b}_y$, $\vec{\nabla}F$ can be expressed as shown right.

$$\begin{aligned} \vec{\nabla}F &\underset{(6.12)}{=} \frac{\partial F}{\partial x}\hat{b}_x + \frac{\partial F}{\partial y}\hat{b}_y \\ &= 2\,x\,\hat{b}_x + 2\,y\,\hat{b}_y \end{aligned}$$

This gradient $\vec{\nabla}F$ calculates an outward normal vector $\vec{n}$ at point Q and a vector $\vec{t}$ tangent to the circle at point Q (directed as shown in the figure).

$$\begin{aligned} \vec{n} &= x\,\hat{b}_x + y\,\hat{b}_y \\ \vec{t} &= \hat{b}_z \times \vec{n} = -y\,\hat{b}_x + x\,\hat{b}_y \end{aligned}$$

Note: Homework 3.13 calculates the normal and tangent for an ellipse.



Gradients are used for analyzing a dynamic celt (rattleback) at www.MotionGenesis.com (GettingStarted link)

6.12 Optional**: Proofs

Proof of vector dot-product derivative property

The proof of the first equation in Section 6.6, i.e., the ***vector dot-product derivative property*** starts by expressing the arbitrary vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in terms of an arbitrary set of orthogonal unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ fixed in an arbitrary reference frame (or rigid basis) A as[5]

$$\vec{\mathbf{u}} = u_x\,\widehat{\mathbf{a}}_x + u_y\,\widehat{\mathbf{a}}_y + u_z\,\widehat{\mathbf{a}}_z \qquad \vec{\mathbf{v}} = v_x\,\widehat{\mathbf{a}}_x + v_y\,\widehat{\mathbf{a}}_y + v_z\,\widehat{\mathbf{a}}_z \tag{13}$$

Forming the scalar quantity $\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}$ and differentiating with respect to the scalar variable t, one finds

$$\begin{aligned} \vec{\mathbf{u}}\cdot\vec{\mathbf{v}} &\underset{(13)}{=} u_x\,v_x + u_y\,v_y + u_z\,v_z \\ \frac{d\,(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})}{dt} &= \dot{u}_x\,v_x + u_x\,\dot{v}_x + \dot{u}_y\,v_y + u_y\,\dot{v}_y + \dot{u}_z\,v_z + u_z\,\dot{v}_z \end{aligned} \tag{14}$$

The next step is to form the right-hand side of the equation being proved, i.e., ${}^{A}\frac{d\vec{\mathbf{u}}}{dt}\cdot\vec{\mathbf{v}} + \vec{\mathbf{u}}\cdot{}^{A}\frac{d\vec{\mathbf{v}}}{dt}$, as

$$\begin{aligned} {}^{A}\frac{d\vec{\mathbf{u}}}{dt} &\underset{(13)}{\triangleq} \dot{u}_x\,\widehat{\mathbf{a}}_x + \dot{u}_y\,\widehat{\mathbf{a}}_y + \dot{u}_z\,\widehat{\mathbf{a}}_z & \qquad {}^{A}\frac{d\vec{\mathbf{u}}}{dt}\cdot\vec{\mathbf{v}} &\underset{(13)}{=} \dot{u}_x\,v_x + \dot{u}_y\,v_y + \dot{u}_z\,v_z \\ {}^{A}\frac{d\vec{\mathbf{v}}}{dt} &\underset{(13)}{\triangleq} \dot{v}_x\,\widehat{\mathbf{a}}_x + \dot{v}_y\,\widehat{\mathbf{a}}_y + \dot{v}_z\,\widehat{\mathbf{a}}_z & \qquad \vec{\mathbf{u}}\cdot{}^{A}\frac{d\vec{\mathbf{v}}}{dt} &\underset{(13)}{=} u_x\,\dot{v}_x + u_y\,\dot{v}_y + u_z\,\dot{v}_z \end{aligned} \tag{15}$$

Combining the two right-most equations in the previous set of equations gives

$${}^{A}\frac{d\vec{\mathbf{u}}}{dt}\cdot\vec{\mathbf{v}} + \vec{\mathbf{u}}\cdot{}^{A}\frac{d\vec{\mathbf{v}}}{dt} \underset{(15)}{=} \dot{u}_x\,v_x + \dot{u}_y\,v_y + \dot{u}_z\,v_z + u_x\,\dot{v}_x + u_y\,\dot{v}_y + u_z\,\dot{v}_z \tag{16}$$

This proof concludes by viewing the equivalence displayed in equations (14) and (16), and recalling that $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ are arbitrary vectors fixed in an arbitrary reference frame (or rigid basis) A. Hence, for arbitrary reference frames (or rigid bases) A and/or B,

$$\boxed{\frac{d(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})}{dt} = {}^{A}\frac{d\,\vec{\mathbf{u}}}{dt}\cdot\vec{\mathbf{v}} + \vec{\mathbf{u}}\cdot{}^{A}\frac{d\,\vec{\mathbf{v}}}{dt} = {}^{B}\frac{d\,\vec{\mathbf{u}}}{dt}\cdot\vec{\mathbf{v}} + \vec{\mathbf{u}}\cdot{}^{B}\frac{d\,\vec{\mathbf{v}}}{dt}}$$

Note: The ***vector dot-product derivative property*** is used to prove the uniqueness and existence of the ***golden rule for vector differentiation*** [equation (7.1)] in Section 7.5.1.

[5]The proof of the ***vector dot-product derivative property*** in Section 6.12 does not require, but is greatly simplified, with orthogonal unit vectors. To do the same proof with non-orthogonal unit vectors $\widehat{\mathbf{a}}_1$, $\widehat{\mathbf{a}}_2$, $\widehat{\mathbf{a}}_3$, one must write $\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}$ in terms of $\widehat{\mathbf{a}}_1\cdot\widehat{\mathbf{a}}_2$, $\widehat{\mathbf{a}}_1\cdot\widehat{\mathbf{a}}_3$, and $\widehat{\mathbf{a}}_2\cdot\widehat{\mathbf{a}}_3$ (which are the cosines of angles between the unit vectors) and then note that for a rigid basis A, the angles between these unit vectors are constant - and hence $\frac{d\,(\widehat{\mathbf{a}}_i\cdot\widehat{\mathbf{a}}_j)}{dt} = 0 \ \ (i,j=1,2,3)$.

Chapter 7

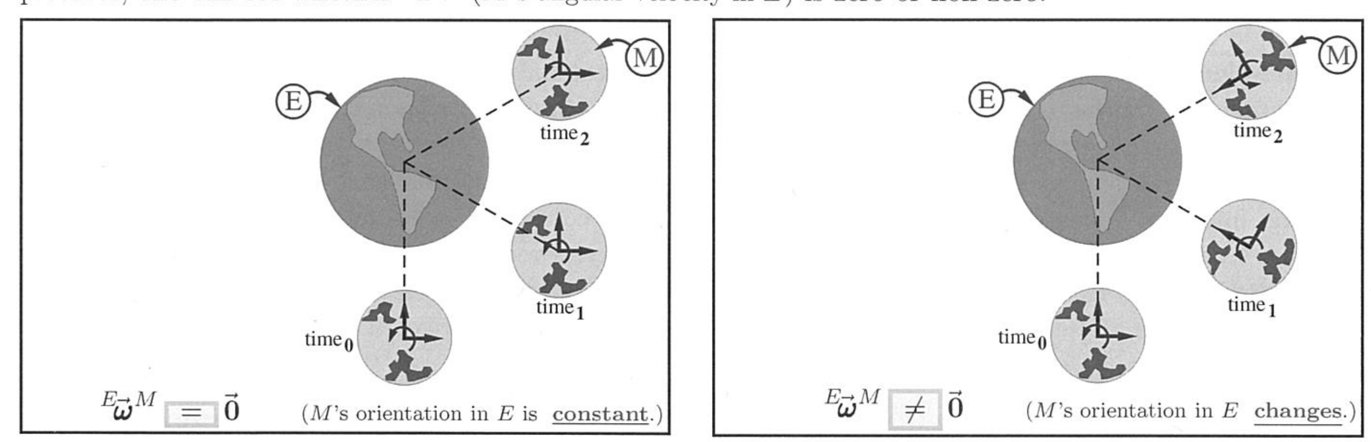
Angular velocity & angular acceleration

Important formulas for angular velocity and angular acceleration. (See examples in Hw 6)

Golden rule for vector differentiation	${}^{A}\frac{d\vec{\mathbf{v}}}{dt} \underset{(7.1)}{=} {}^{B}\frac{d\vec{\mathbf{v}}}{dt} + {}^{A}\vec{\boldsymbol{\omega}}^{B}\times\vec{\mathbf{v}}$
Simple angular velocity	${}^{A}\vec{\boldsymbol{\omega}}^{B} \underset{(7.5)}{=} \pm\,\dot{\theta}\boldsymbol{\lambda}$ when $\boldsymbol{\lambda}$ is fixed in both A and B
Angular velocity negative property	${}^{A}\vec{\boldsymbol{\omega}}^{B} \underset{(7.6)}{=} -{}^{B}\vec{\boldsymbol{\omega}}^{A}$
Angular velocity addition theorem	${}^{A}\vec{\boldsymbol{\omega}}^{D} \underset{(7.7)}{=} {}^{A}\vec{\boldsymbol{\omega}}^{B} + {}^{B}\vec{\boldsymbol{\omega}}^{C} + {}^{C}\vec{\boldsymbol{\omega}}^{D}$
Angular velocity and arbitrary vectors	See equation (2)
Angular velocity and basis vectors	${}^{A}\vec{\boldsymbol{\omega}}^{B} \underset{(7.3)}{=} ({}^{A}\frac{d\widehat{\mathbf{b}}_y}{dt}\cdot\widehat{\mathbf{b}}_z)\,\widehat{\mathbf{b}}_x + ({}^{A}\frac{d\widehat{\mathbf{b}}_z}{dt}\cdot\widehat{\mathbf{b}}_x)\,\widehat{\mathbf{b}}_y + ({}^{A}\frac{d\widehat{\mathbf{b}}_x}{dt}\cdot\widehat{\mathbf{b}}_y)\,\widehat{\mathbf{b}}_z$
Angular velocity and rotation matrices	See equation (4)
Partial angular velocity (Kane's method)	See Section 26.2 (also for virtual angular displacement)
B's angular acceleration in A	${}^{A}\vec{\boldsymbol{\alpha}}^{B} \underset{(7.8)}{\triangleq} \frac{{}^{A}d\,{}^{A}\vec{\boldsymbol{\omega}}^{B}}{dt} \underset{(7.9)}{=} \frac{{}^{B}d\,{}^{A}\vec{\boldsymbol{\omega}}^{B}}{dt}$
Angular acceleration addition theorem	${}^{A}\vec{\boldsymbol{\alpha}}^{C} \underset{(7.10)}{=} {}^{A}\vec{\boldsymbol{\alpha}}^{B} + {}^{B}\vec{\boldsymbol{\alpha}}^{C} + {}^{A}\vec{\boldsymbol{\omega}}^{B}\times{}^{B}\vec{\boldsymbol{\omega}}^{C}$
Angular acceleration negative property	${}^{A}\vec{\boldsymbol{\alpha}}^{B} \underset{(7.11)}{=} -{}^{B}\vec{\boldsymbol{\alpha}}^{A}$

7.1 Angular velocity concepts: Moon and Earth celestial systems

Each of the two pictures below depict a moon M in counter-clockwise orbit about a planet E. From the pictures, one can see whether ${}^{E}\vec{\boldsymbol{\omega}}^{M}$ (M's angular velocity in E) is zero or non-zero.[1]



[1]Although the left-moon's angular velocity is $\vec{0}$, one can construct a rigid vector basis B so that ${}^{E}\vec{\boldsymbol{\omega}}^{B} \neq \vec{0}$ by using an Earth-to-Moon pointing vector and a vector perpendicular to the orbital plane. Note: A particle only translates whereas a rigid body can translate and rotate. A particle can translate around Earth - but the particle is not "rotating" (particles do not possess orientation). Conversely, a rigid body can translate around Earth and its orientation may change.