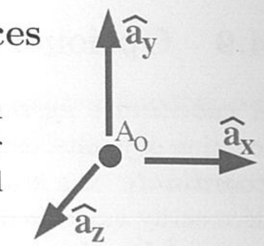
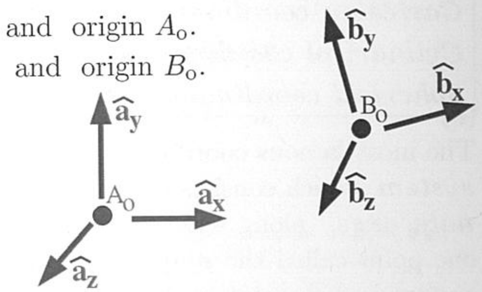
4.10 Optional**: Rigid frame and transformation matrices



A ***rigid frame*** is the combination of a ***rigid vector basis*** and an ***origin*** point. A rigid frame is a useful measuring device for kinematics and multibody dynamics. For example, the figure to the right shows a rigid frame **A** having right-handed orthogonal unit vectors $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and an origin A_o.

Optional**: Transformation matrices (for robotics, graphics, ...)

The following figure shows two ***rigid frames*** A and B.
Rigid frame A has right-handed orthogonal unit vectors $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and origin A_o.
Rigid frame B has right-handed orthogonal unit vectors $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ and origin B_o.

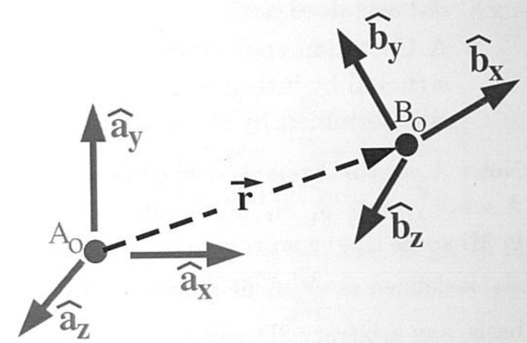


The 4×4 ***transformation matrix*** ${}^A T^B$ stores the orientation of A and B in a 3×3 ${}^a R^b$ ***rotation matrix*** (described in Chapter 5) and stores the $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ measures of B_o's position vector from A_o in a 3×1 sub-matrix as shown below.

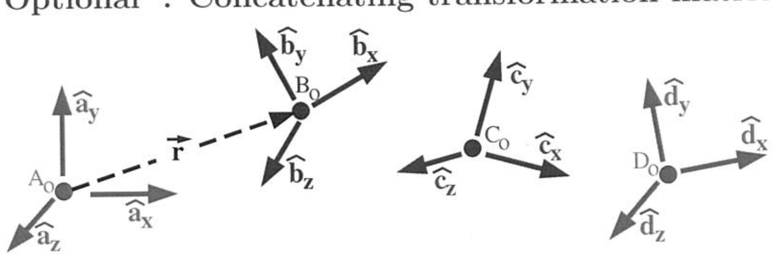
$$
{}^A T^B \triangleq \left[\begin{array}{ccc|c} & {}^a R^b & & \left[\vec{r}^{\,B_o/A_o}\right]_a \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} \hat{a}_x \cdot \hat{b}_x & \hat{a}_x \cdot \hat{b}_y & \hat{a}_x \cdot \hat{b}_z & \hat{a}_x \cdot \vec{r}^{\,B_o/A_o} \\ \hat{a}_y \cdot \hat{b}_x & \hat{a}_y \cdot \hat{b}_y & \hat{a}_y \cdot \hat{b}_z & \hat{a}_y \cdot \vec{r}^{\,B_o/A_o} \\ \hat{a}_z \cdot \hat{b}_x & \hat{a}_z \cdot \hat{b}_y & \hat{a}_z \cdot \hat{b}_z & \hat{a}_z \cdot \vec{r}^{\,B_o/A_o} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \quad (4)
$$

Numerical values representative of the schematic to the right can be stored in a 4×4 ***transformation table***, e.g., as

${}^A T^B$	$\hat{b}_x$	$\hat{b}_y$	$\hat{b}_z$	$\vec{r}^{\,B_o/A_o}$
$\hat{a}_x$	0.6958	-0.7165	0.0500	4.2
$\hat{a}_y$	0.7103	0.6968	0.0100	3.3
$\hat{a}_z$	-0.1063	-0.0339	0.9938	-1.2
	0	0	0	1



Optional**: Concatenating transformation matrices for rigid frames A, B, C, D



The 4×4 ***transformation matrix*** ${}^A T^D$ relates the orientation and position of rigid frames A and D and can be computed by matrix multiplication as shown below.

$$
{}^A T^D = {}^A T^B * {}^B T^C * {}^C T^D
$$

Optional**: Inverse of a transformation matrix

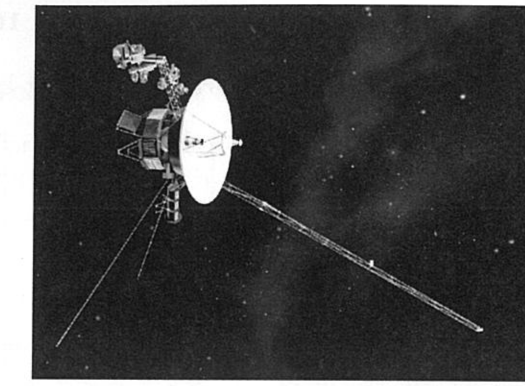
Since rotation matrices are orthogonal, ${}^b R^a$ has the special property ${}^b R^a = \left({}^a R^b\right)^{-1} = \left({}^a R^b\right)^T$.

Using this special rotation matrix property, the inverse of a transformation matrix can be efficiently calculated as shown right (more efficient and accurate than inverting a 4×4 matrix).

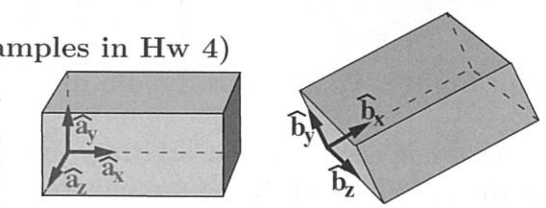
$$
{}^B T^A = \left({}^A T^B\right)^{-1} = \left[\begin{array}{ccc|c} & {}^b R^a & & -{}^b R^a * \left[\vec{r}^{\,B_o/A_o}\right]_a \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

Chapter 5

Rotation matrices I



Rotation matrices and vector bases (see examples in Hw 4)



The figure to the right shows two ***right-handed, unitary orthogonal bases*** a and b whose basis vectors are $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$, respectively. ***Orientation*** between a and b can be described with a 3 × 3 ***rotation matrix*** ${}^a R^b$, whose elements ${}^a R^b_{ij}$ $(i, j = x, y, z)$ are defined in equation (1).

$$
{}^a R^b_{ij} \triangleq \hat{a}_i \cdot \hat{b}_j \overset{(2.2)}{=} \cos\angle(\hat{a}_i, \hat{b}_j) \qquad (i, j = x, y, z) \quad (1)
$$

The definition of dot-product in equation (2.2) shows ${}^a R^b_{ij}$ is equal to the cosine of the angle between $\hat{a}_i$ and $\hat{b}_j$.[a]

[a] A rotation matrix is also called a ***direction cosine matrix*** whose elements are ***direction cosines***.

Section 5.4.1 gives an example of calculating angles from a rotation matrix. Chapter 8 shows the orientation of a in b can be characterized in various ways, including Euler angles, Euler parameters (quaternions), and Rodrigues parameters.

A ***rotation matrix*** R is an orthogonal matrix, which means the transpose of R is equal to the inverse of R, i.e., $R^T = R^{-1}$.
Section 5.4.2 gives an example of calculating the inverse of a rotation matrix.

$$
{}^b R^a = \left({}^a R^b\right)^{-1} = \left({}^a R^b\right)^T \quad (2)
$$

Since rotation matrices are orthogonal, it is convenient to store the orientation information in a ***rotation table*** that can be read horizontally or vertically.[a]

${}^a R^b$	$\hat{b}_x$	$\hat{b}_y$	$\hat{b}_z$
$\hat{a}_x$	$\hat{a}_x \cdot \hat{b}_x$	$\hat{a}_x \cdot \hat{b}_y$	$\hat{a}_x \cdot \hat{b}_z$
$\hat{a}_y$	$\hat{a}_y \cdot \hat{b}_x$	$\hat{a}_y \cdot \hat{b}_y$	$\hat{a}_y \cdot \hat{b}_z$
$\hat{a}_z$	$\hat{a}_z \cdot \hat{b}_x$	$\hat{a}_z \cdot \hat{b}_y$	$\hat{a}_z \cdot \hat{b}_z$

(3)

[a] Section 5.7.1 gives the proof of equation (2). If one or both of the bases are non-orthogonal, a rotation matrix (not a rotation table) is used because the inverse of a non-orthogonal matrix is not its transpose.

The rotation matrix ${}^a R^d$ can be formed by successive matrix multiplication of the ${}^a R^b$, ${}^b R^c$, and ${}^c R^d$ rotation matrices.
Section 5.5.4 gives an example of successive matrix multiplication.

$$
{}^a R^d = {}^a R^b * {}^b R^c * {}^c R^d \quad (4)
$$

5.1 Uses for the rotation matrix ${}^a R^b$ (for geometry, statics, motion analysis, stress ...)

- Finding the **dot-product** between the unit vectors $\hat{a}_i$ and $\hat{b}_j$, $(i, j = x, y, z)$
- Calculating the **angle** between the unit vectors $\hat{a}_i$ and $\hat{b}_j$, $(i, j = x, y, z)$
- Calculating the **dot-product** or **cross-product** between a vector $\vec{v}$ and a vector $\vec{w}$, each of which may be expressed in terms of $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and/or $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ (e.g., see Section 5.4.3).
- **Expressing** a vector written in terms of $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ in terms of $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ (or vice-versa)
- Calculating other rotation matrices e.g., ${}^a R^d = {}^a R^b * {}^b R^c * {}^c R^d$ [see equation (6)].
- Relating the column matrix representation of a vector $\vec{v}$ expressed in a vector basis a to its column matrix representation in a vector basis b, e.g., $[\vec{v}]_a = {}^a R^b \, [\vec{v}]_b$

5.2 Rotation matrices and matrix multiplication

Two ***orthogonal*** (or *non-orthogonal*) unit bases are related with **column matrices** of unit vectors as

$$\begin{bmatrix}\hat{a}_x\\ \hat{a}_y\\ \hat{a}_z\end{bmatrix} = {}^aR^b \begin{bmatrix}\hat{b}_x\\ \hat{b}_y\\ \hat{b}_z\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}\hat{b}_x\\ \hat{b}_y\\ \hat{b}_z\end{bmatrix} = {}^bR^a \begin{bmatrix}\hat{a}_x\\ \hat{a}_y\\ \hat{a}_z\end{bmatrix} \quad \text{where} \quad {}^bR^a = \left({}^aR^b\right)^{-1}$$

Alternately, transposing the previous equation shows **row matrices** of orthogonal unit vectors are related by

$$\begin{bmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\end{bmatrix} = \begin{bmatrix}\hat{b}_x & \hat{b}_y & \hat{b}_z\end{bmatrix} {}^bR^a \quad \text{or} \quad \begin{bmatrix}\hat{b}_x & \hat{b}_y & \hat{b}_z\end{bmatrix} = \begin{bmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\end{bmatrix} {}^aR^b \tag{5}$$

Optional**: Vector and dyadic measures and rotation matrices

As shown in Section 4.6, an arbitrary vector $\vec{v}$ can be expressed in terms of the right-handed orthogonal unit vectors $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and the scalars v_x, v_y, v_z as shown to the right. Alternately, $\vec{v}$ can be expressed in terms of the right-handed orthogonal unit vectors $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ and the scalars $\bar{v}_x$, $\bar{v}_y$, $\bar{v}_z$. As proved in Section 5.7.2, the scalars $\bar{v}_x$, $\bar{v}_y$, $\bar{v}_z$ and v_x, v_y, v_z are related as shown in equation (6).

$$\vec{v} = v_x\,\hat{a}_x + v_y\,\hat{a}_y + v_z\,\hat{a}_z$$
$$\vec{v} = \bar{v}_x\,\hat{b}_x + \bar{v}_y\,\hat{b}_y + \bar{v}_z\,\hat{b}_z$$
$$\begin{bmatrix}\bar{v}_x\\ \bar{v}_y\\ \bar{v}_z\end{bmatrix}_b = {}^bR^a \begin{bmatrix}v_x\\ v_y\\ v_z\end{bmatrix}_a \tag{6}$$

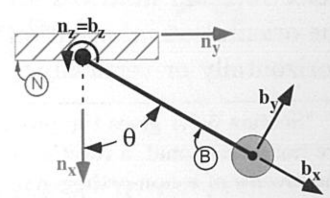
A more compact form of equation (6) equates the column matrix representation of $\vec{v}$ expressed in basis a to the column matrix representation of $\vec{v}$ expressed in basis b as shown in equation (7).

$$\boxed{[\vec{v}]_b = {}^bR^a\,[\vec{v}]_a} \tag{7}$$

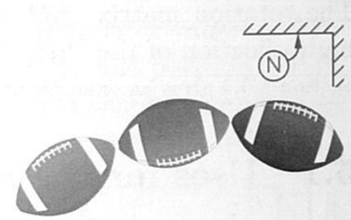
Similarly, equation (8) relates the 3×3 matrix representations of a dyadic $\vec{\vec{D}}$ express in the a and b bases (proved in Section 16.2).

$$\boxed{\left[\vec{\vec{D}}\right]_b = {}^bR^a \left[\vec{\vec{D}}\right]_a {}^aR^b} \tag{8}$$

5.3 Rotation matrices - who cares?



In 2D (***two dimensional***) analysis, the orientation of a rigid body B in a reference frame N **can** be characterized with a **single angle**, e.g., the "pendulum" angle θ shown to the right. Since B rotates in a plane perpendicular to $\hat{n}_z = \hat{b}_z$, θ is defined as the angle from $\hat{n}_x$ to $\hat{b}_x$ with $+\hat{b}_z$ sense.



In 3D (***three dimensional***) analysis, the orientation of a rigid body B (e.g., a spiraling, wobbling, football) in a reference frame N (e.g., a stadium) **cannot** be characterized by a single angle.[a] One convenient way to measure the football's orientation in the stadium is with a rotation matrix.

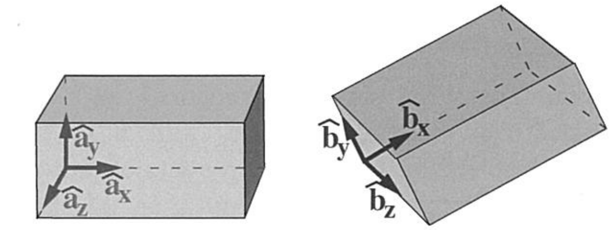
[a]In 3D analysis, a rigid body can be oriented with: a single angle *and* a vector, or four Euler parameters (quaternion), or three Euler angles, or a rotation matrix, or ...

5.4 Simple rotation matrix examples

5.4.1 Example: Calculating angles between unit vectors $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$

A ***rotation table*** stores ***dot-products*** between unit vectors and makes it easy to calculate angles between unit vectors. For example, ${}^aR^b$ stores $\hat{a}_x \cdot \hat{b}_z = 0.258819$ in the $\hat{a}_x$ row and $\hat{b}_z$ column. Using the definition of dot-product in equation (2.2), the angle between $\hat{a}_x$ and $\hat{b}_z$ can be calculated as shown below.

${}^aR^b$	$\hat{b}_x$	$\hat{b}_y$	$\hat{b}_z$
$\hat{a}_x$	0.9622502	-0.08418598	0.258819
$\hat{a}_y$	0.1700841	0.9284017	-0.3303661
$\hat{a}_z$	-0.2124758	0.3619158	0.9076734



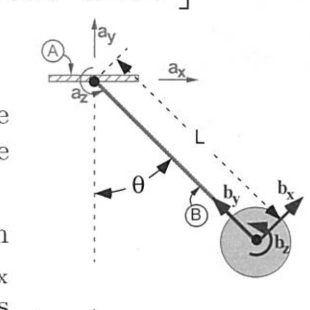
$$\angle(\hat{a}_x, \hat{b}_z) = \text{acos}(0.258819) = 75°$$

5.4.2 Example: Calculation of rotation matrix inverse

The following rotation matrix R relates two right-handed, orthogonal, unitary bases. Since a rotation matrix is orthogonal, its inverse can be written down by inspection.

$$R = \begin{bmatrix} 0.433 & 0.0580 & 0.8995\\ -0.25 & 0.9665 & 0.0580\\ -0.866 & -0.25 & 0.4330 \end{bmatrix} \quad \Rightarrow \quad R^{-1} = \begin{bmatrix} 0.433 & -0.25 & -0.866\\ 0.0580 & 0.9665 & -0.25\\ 0.8995 & 0.0580 & 0.4330 \end{bmatrix}$$

5.4.3 Example: Forming and using a rotation matrix



The figure to the right shows a rod B connected to a fixed support A by a revolute joint. Right-handed sets of orthogonal unit vectors $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$, are fixed in A and B, respectively. The ${}^bR^a$ rotation matrix is given to the right.

This example shows how to use a rotation matrix, e.g., to ***express a vector*** in another basis and perform dot-products and cross-products. For example, the $\hat{b}_x$ row of the ${}^bR^a$ rotation table allows $\hat{b}_x$ to be expressed in terms of $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ as

${}^bR^a$	$\hat{a}_x$	$\hat{a}_y$	$\hat{a}_z$
$\hat{b}_x$	cos(θ)	sin(θ)	0
$\hat{b}_y$	-sin(θ)	cos(θ)	0
$\hat{b}_z$	0	0	1

$$\hat{b}_x = \cos(\theta)\,\hat{a}_x + \sin(\theta)\,\hat{a}_y$$

The $\hat{a}_x$ column of ${}^bR^a$ allows $\hat{a}_x$ to be expressed in terms of $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ as

$$\hat{a}_x = \cos(\theta)\,\hat{b}_x - \sin(\theta)\,\hat{b}_y$$

The **dot-product** $\hat{b}_y \cdot \hat{a}_x$ is the element of ${}^bR^a$ in the $\hat{b}_y$ row and $\hat{a}_x$ column, i.e., $\hat{b}_y \cdot \hat{a}_x = \text{-}\sin(\theta)$. A more complicated **dot-product** example computes

$$(\hat{a}_x + 2\,\hat{a}_y) \cdot \left(x\,\hat{b}_x + y\,\hat{b}_y\right) = x\,(\hat{a}_x \cdot \hat{b}_x) + y\,(\hat{a}_x \cdot \hat{b}_y) + 2\,x\,(\hat{a}_y \cdot \hat{b}_x) + 2\,y\,(\hat{a}_y \cdot \hat{b}_y)$$
$$= x\,[\cos(\theta)] + y\,[\text{-}\sin(\theta)] + 2\,x\,[\sin(\theta)] + 2\,y\,[\cos(\theta)]$$

An example of doing a mixed-basis **cross-product** is

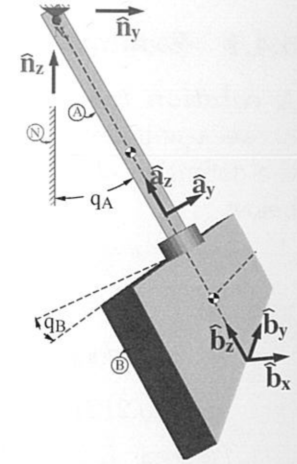
$$\hat{b}_x \times (x\,\hat{a}_x + y\,\hat{a}_y) = [\cos(\theta)\,\hat{a}_x + \sin(\theta)\,\hat{a}_y] \times (x\,\hat{a}_x + y\,\hat{a}_y) = [y\,\cos(\theta) - x\,\sin(\theta)]\,\hat{a}_z$$

A more complicated **cross-product** example computes

$$(\hat{a}_x + 2\,\hat{a}_y) \times \left(x\,\hat{b}_x + y\,\hat{b}_y\right) = \left\{[\cos(\theta) + 2\,\sin(\theta)]\,\hat{b}_x + [2\,\cos(\theta) - \sin(\theta)]\,\hat{b}_y\right\} \times \left(x\,\hat{b}_x + y\,\hat{b}_y\right)$$
$$= \{y\,[\cos(\theta) + 2\,\sin(\theta)] - x\,[2\,\cos(\theta) - \sin(\theta)]\}\,\hat{b}_z$$

5.5 Example: Rotation matrices & matrix multiplication

Shown to the right is a rigid rod A connected to a fixed support N by a revolute joint whose horizontal axis is parallel to the unit vectors $\hat{n}_x = \hat{a}_x$. Rod A's orientation in N is characterized by the right-handed rotation $q_A\,\hat{a}_x$.



Rigid plate B is connected to rod A by another revolute joint whose axis is parallel to the unit vectors $\hat{a}_z = \hat{b}_z$ (B can rotate freely about A's long axis). Plate B's orientation in N is characterized by the right-handed rotation $q_B\,\hat{b}_z$.

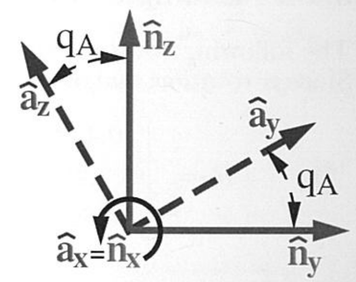
Note: The B-to-A revolute joint is perpendicular to the A-to-N revolute joint.

There are three sets of orthogonal unit vectors, fixed in N, A, and B, respectively, namely $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$ and $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$.
This problem relates these three sets of unit vectors with rotation matrices.[a]

[a] Simple rotation matrices are formed using the definitions of sine and cosine (SohCahToa).

5.5.1 Example: Forming the simple rotation matrix ${}^{a}R^{n}$

The rotation matrix ${}^{a}R^{n}$ relating the right-handed sets of orthogonal unit vectors $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$ shown to the right is a "***simple rotation matrix***" because $\hat{a}_x = \hat{n}_x$ for the duration of A's rotation in N.



To calculate ${}^{a}R^{n}$, it is helpful to **redraw these vectors** in the geometrically suggestive way shown to the right. After using the definitions of sine and cosine to express each of $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ in terms of $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$ one can form the ${}^{a}R^{n}$ rotation table as shown below.

$$\begin{aligned}\hat{a}_x &= \hat{n}_x\\ \hat{a}_y &= \cos(q_A)\,\hat{n}_y + \sin(q_A)\,\hat{n}_z\\ \hat{a}_z &= \text{-}\sin(q_A)\,\hat{n}_y + \cos(q_A)\,\hat{n}_z\end{aligned} \quad\Rightarrow$$

${}^{a}R^{n}$	$\hat{n}_x$	$\hat{n}_y$	$\hat{n}_z$
$\hat{a}_x$	1	0	0
$\hat{a}_y$	0	$\cos(q_A)$	$\sin(q_A)$
$\hat{a}_z$	0	$\text{-}\sin(q_A)$	$\cos(q_A)$

The rotation **table** makes it easy to form ${}^{a}R^{n}$, the rotation **matrix** relating $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ to $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$.
Shown next, the rotation matrix ${}^{n}R^{a}$ is ***quickly and accurately calculated*** by the ***transpose*** of ${}^{a}R^{n}$.

$$\begin{bmatrix}\hat{a}_x\\ \hat{a}_y\\ \hat{a}_z\end{bmatrix} = {}^{a}R^{n}\begin{bmatrix}\hat{n}_x\\ \hat{n}_y\\ \hat{n}_z\end{bmatrix} \quad\text{or}\quad \begin{bmatrix}\hat{a}_x\\ \hat{a}_y\\ \hat{a}_z\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos(q_A) & \sin(q_A)\\ 0 & \text{-}\sin(q_A) & \cos(q_A)\end{bmatrix}\begin{bmatrix}\hat{n}_x\\ \hat{n}_y\\ \hat{n}_z\end{bmatrix}$$

$$\begin{bmatrix}\hat{n}_x\\ \hat{n}_y\\ \hat{n}_z\end{bmatrix} = {}^{n}R^{a}\begin{bmatrix}\hat{a}_x\\ \hat{a}_y\\ \hat{a}_z\end{bmatrix} \quad\text{or}\quad \begin{bmatrix}\hat{n}_x\\ \hat{n}_y\\ \hat{n}_z\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos(q_A) & \text{-}\sin(q_A)\\ 0 & \sin(q_A) & \cos(q_A)\end{bmatrix}\begin{bmatrix}\hat{a}_x\\ \hat{a}_y\\ \hat{a}_z\end{bmatrix}$$

5.5.2 Hug rule (a pattern for quickly forming simple rotation matrices)

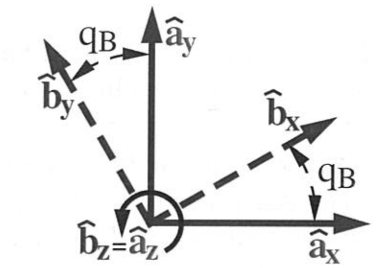
When: Two sets of orthogonal unit vectors $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ and $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ are initially set so $\hat{a}_i = \hat{b}_i$ ($i = x, y, z$) and b undergoes a simple rotation relative to a about one of $\hat{b}_i = \hat{a}_i$ by an angle θ.
Then: The $\hat{b}_i$ row and $\hat{a}_i$ column contain only 1 or 0 and the remaining elements of the rotation table have the pattern show below right.

The $\pm$ sign is **+** if the unit vector is "hugged" ($\hat{b}_x$ is between $\hat{a}_x$ and $\hat{a}_y$) or **-** when the unit vector is "left out in the cold" ($\hat{b}_y$ is not between $\hat{a}_x$ and $\hat{a}_y$). [***Hug rule*** analogy courtesy of Dr. Mandy Koop].

$\cos(\theta)$	$\pm\sin(\theta)$
$\pm\sin(\theta)$	$\cos(\theta)$

5.5.3 Example: Forming the simple rotation matrix ${}^{b}R^{a}$

The rotation matrix ${}^{b}R^{a}$ relating the right-handed sets of orthogonal unit vectors $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ and $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ shown to the right is a "***simple rotation matrix***" because $\hat{b}_z = \hat{a}_z$ for the duration of b's rotation in a.



To calculate ${}^{b}R^{a}$, it is helpful to **redraw these vectors** in the geometrically suggestive way shown to the right. After using the definitions of sine and cosine to express each of $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ in terms of $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$, one can form the ${}^{b}R^{a}$ rotation table as shown below.

$$\begin{aligned}\hat{b}_x &= \cos(q_B)\,\hat{a}_x + \sin(q_B)\,\hat{a}_y\\ \hat{b}_y &= \text{-}\sin(q_B)\,\hat{a}_x + \cos(q_B)\,\hat{a}_y\\ \hat{b}_z &= \hat{a}_z\end{aligned} \quad\Rightarrow$$

${}^{b}R^{a}$	$\hat{a}_x$	$\hat{a}_y$	$\hat{a}_z$
$\hat{b}_x$	$\cos(q_B)$	$\sin(q_B)$	0
$\hat{b}_y$	$\text{-}\sin(q_B)$	$\cos(q_B)$	0
$\hat{b}_z$	0	0	1

5.5.4 Example: Rotation matrix multiplication to form ${}^{b}R^{n} = {}^{b}R^{a} * {}^{a}R^{n}$

The rotation matrix ${}^{b}R^{n}$ relating $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ and $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$ is formed as ${}^{b}R^{n} \underset{(4)}{=} {}^{b}R^{a} * {}^{a}R^{n}$.

$${}^{b}R^{n} = \begin{bmatrix}\cos(q_B) & \sin(q_B) & 0\\ \text{-}\sin(q_B) & \cos(q_B) & 0\\ 0 & 0 & 1\end{bmatrix} * \begin{bmatrix}1 & 0 & 0\\ 0 & \cos(q_A) & \sin(q_A)\\ 0 & \text{-}\sin(q_A) & \cos(q_A)\end{bmatrix} = \begin{bmatrix}\cos(q_B) & \sin(q_B)\cos(q_A) & \sin(q_B)\sin(q_A)\\ \text{-}\sin(q_B) & \cos(q_B)\cos(q_A) & \cos(q_B)\sin(q_A)\\ 0 & \text{-}\sin(q_A) & \cos(q_A)\end{bmatrix}$$

The ${}^{b}R^{n}$ rotation table (shown right) is copied from its associated rotation matrix and is the starting point for most relationships between $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ and $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$.
For example, reading the ${}^{b}R^{n}$ rotation table gives

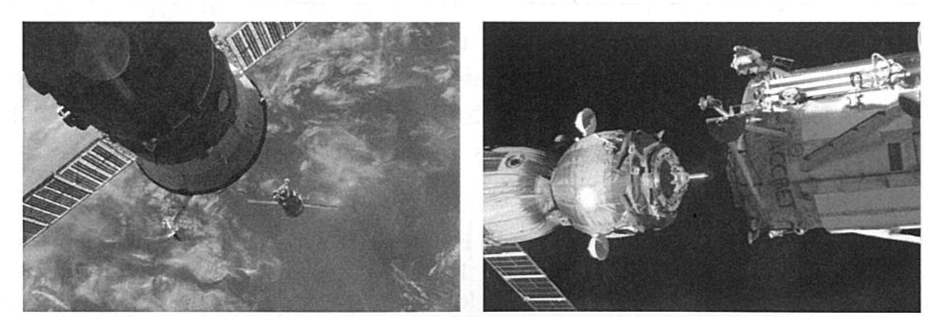
${}^{b}R^{n}$	$\hat{n}_x$	$\hat{n}_y$	$\hat{n}_z$
$\hat{b}_x$	$\cos(q_B)$	$\sin(q_B)\cos(q_A)$	$\sin(q_B)\sin(q_A)$
$\hat{b}_y$	$\text{-}\sin(q_B)$	$\cos(q_B)\cos(q_A)$	$\cos(q_B)\sin(q_A)$
$\hat{b}_z$	0	$\text{-}\sin(q_A)$	$\cos(q_A)$

First row: $\hat{b}_x = \cos(q_B)\,\hat{n}_x + \sin(q_B)\cos(q_A)\,\hat{n}_y + \sin(q_B)\sin(q_A)\,\hat{n}_z$

First column: $\hat{n}_x = \cos(q_B)\,\hat{b}_x + \text{-}\sin(q_B)\,\hat{b}_y + 0\,\hat{b}_z$

As shown below, the dot product $\hat{b}_x \cdot \hat{n}_z$ is simply the element in the $\hat{b}_x$ row and $\hat{n}_z$ column of the ${}^{b}R^{n}$ rotation table and the angle between $\hat{b}_x$ and $\hat{n}_z$ can be calculated via the definition of the dot-product.

$$\hat{b}_x \cdot \hat{n}_z = \sin(q_B)\sin(q_A) \qquad \angle(\hat{b}_x, \hat{n}_z) = \text{acos}\,[\sin(q_B)\sin(q_A)]$$

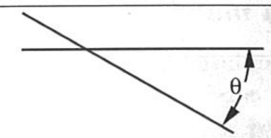


Rotation matrices are used for relative orientation such as the Soyuz spacecraft docking with the international space station, or tele-robotic surgeries with needle/catheter insertion, etc.

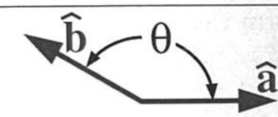
5.6 What is an angle?

The definition of the word "*angle*" is context-sensitive, e.g., depending on whether measurements involve sweep between two lines, two vectors, two vectors and a sense vector, and/or time.[1] [2]

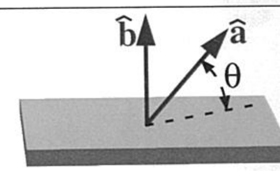
Angle between two lines $(0 \le \theta \le 90°)$
The angle θ between two lines is defined as the smallest angle between all vectors aligned with the lines, hence, $0 \le \theta \le 90°$.



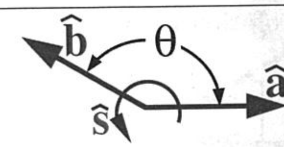
Angle between two vectors $(0 \le \theta \le 180°)$
The angle θ between two vectors $\vec{a}$ and $\vec{b}$ is defined as the smallest angle between $\vec{a}$ and $\vec{b}$, and can be calculated from equation (2.2) as
$\theta = \text{acos}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$ (the acos function returns $0 \le \theta \le 180°$).



Angle between a vector and a plane $(0 \le \theta \le 180°)$
The angle θ between a vector $\vec{a}$ and a plane perpendicular to a vector $\vec{b}$ is $\theta = 90° - \angle(\vec{a}, \vec{b})$, where $\angle(\vec{a}, \vec{b})$ is the angle between $\vec{a}$ and $\vec{b}$.
Alternately, if $\vec{b}$ points oppositely so $\vec{a} \cdot \vec{b} < 0$, $\theta = \angle(\vec{a}, \vec{b}) - 90°$.

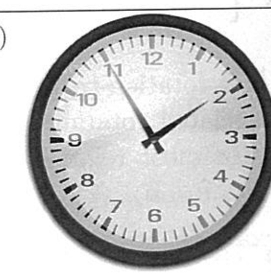


Angle from a vector to a vector with a sense vector $(-180° < \theta \le 180°)$
The angle θ from vector $\vec{a}$ to vector $\vec{b}$ with positive sense about vector $\vec{s}$ is is regarded as positive when $(\vec{a} \times \vec{b}) \cdot \vec{s} \ge 0$ and negative when $(\vec{a} \times \vec{b}) \cdot \vec{s} < 0$, hence $-180° < \theta \le 180°$, calculated as $\theta = \pm\, \text{acos}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$.



Angle between two vectors with a sense vector and time $(-\infty < \theta < +\infty)$
The angle θ from vector $\vec{a}$ to vector $\vec{b}$ with positive sense about vector $\vec{s}$ with time can be calculated using *wrap* and may have values $-\infty < \theta < +\infty$.

Note: With only two lines or two vectors, an angle is an inherently non-negative quantity. An angle may be *regarded* as negative when one associates a positive sense with its value, e.g., associating a positive clockwise sense.



In certain applications, names are given to various angles as shown in the following table.

Application	Names of angles
Aerospace and automotive	roll, pitch, yaw or heading, attitude, bank
Gait analysis (hip)	rotation, obliquity, torsion
Knee analysis	flexion/extension, internal/external rotation, adduction/abduction
Surveying and astronomy	inclination/declination, ascension, azimuth, elevation, grade, pitch
Spinning rigid body	nutation, spin, precession, libration, wobble
Diving	somersault, tilt, twist

There is no precise universally-agreed definition for these angles, particularly when all three angles are non-zero and large. For example, medical doctors, physiologists, physical therapists, and lab technicians have loosely used terms like flexion and extension for centuries. Recent biomechanical studies rely on accurate measurements of these angles - and requires more precise definition (see Section 8.4).

[1] Height provides a useful analogy to angle. Usually, height is inherently non-negative, e.g., a person's height is a positive quantity. However, one may report height above seal-level as -10 m by implying an upward positive sense. Similarly, angles may be negative by providing a positive sense. Historically, angles (e.g., used by the ancient Greeks) predate negative numbers (in widespread use by Europeans in 1700 A.D.) by thousands of years.

[2] For example, the "*dihedral angle*" between two planes is the angle between the normals to the two planes.

5.7 Optional**: Proofs

5.7.1 Proof that a rotation matrix is orthonormal

The ${}^{a}R^{b}$ rotation table shown to the right relates two sets of right-handed, orthogonal, unit vectors, namely, $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$.

${}^{a}R^{b}$	$\hat{b}_x$	$\hat{b}_y$	$\hat{b}_z$
$\hat{a}_x$	R_{xx}	R_{xy}	R_{xz}
$\hat{a}_y$	R_{yx}	R_{yy}	R_{yz}
$\hat{a}_z$	R_{zx}	R_{zy}	R_{zz}

Since $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ are orthogonal unit vectors, one knows

$$\hat{a}_i \cdot \hat{a}_i = 1 \quad (i = x, y, z) \qquad \text{and} \qquad \hat{a}_i \cdot \hat{a}_j = 0 \quad (i \ne j)$$

In view of the ${}^{a}R^{b}$ rotation table, $\hat{a}_i \cdot \hat{a}_j$ can also be written in terms of R_{ij} $(i, j, k = x, y, z)$, as

$$\begin{aligned}
\hat{a}_x \cdot \hat{a}_x &= 1 = R_{xx}^2 + R_{xy}^2 + R_{xz}^2 \\
\hat{a}_x \cdot \hat{a}_y &= 0 = R_{xx}\,R_{yx} + R_{xy}\,R_{yy} + R_{xz}\,R_{yz} \\
\hat{a}_x \cdot \hat{a}_z &= 0 = R_{xx}\,R_{zx} + R_{xy}\,R_{zy} + R_{xz}\,R_{zz} \\
\hat{a}_y \cdot \hat{a}_x &= 0 = R_{yx}\,R_{xx} + R_{yy}\,R_{xy} + R_{yz}\,R_{xz} \\
\hat{a}_y \cdot \hat{a}_y &= 1 = R_{yx}^2 + R_{yy}^2 + R_{yz}^2 \\
&\ldots
\end{aligned}$$

To show ${}^{a}R^{b} * ({}^{a}R^{b})^T$ equals the identity matrix I, multiply the rotation matrix by its transpose, i.e.,

$$\begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix} * \begin{bmatrix} R_{xx} & R_{yx} & R_{zx} \\ R_{xy} & R_{yy} & R_{zy} \\ R_{xz} & R_{yz} & R_{zz} \end{bmatrix} = \begin{bmatrix} R_{xx}^2 + R_{xy}^2 + R_{xz}^2 & R_{xx}\,R_{yx} + R_{xy}\,R_{yy} + R_{xz}\,R_{yz} & \ldots \\ R_{yx}\,R_{xx} + R_{yy}\,R_{xy} + R_{yz}\,R_{xz} & R_{yx}^2 + R_{yy}^2 + R_{yz}^2 & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}$$

The right-most matrix in the previous equation is the identity matrix I as it elements $\hat{a}_i \cdot \hat{a}_j$ $(i, j, k = x, y, z)$ are 1 or 0. Invoking the definition of the matrix inverse concludes the proof of equation (2), i.e.,

$$\boxed{({}^{a}R^{b})^{-1} = ({}^{a}R^{b})^{T}}$$

5.7.2 Proof of relationship between $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ and $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ measures of a vector $\vec{v}$

The proof of equation (6) starts by expressing $\vec{v}$ in terms of $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and using the elements of the ${}^{a}R^{b}$ rotation matrix defined in equations (1) and (3) to re-express each of $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ in terms of $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ as

$$\vec{v} = v_x\,\hat{a}_x + v_y\,\hat{a}_y + v_z\,\hat{a}_z$$

$$\begin{aligned}
\vec{v} \underset{(1,3)}{=} \; & v_x \left({}^{a}R^{b}_{xx}\,\hat{b}_x + {}^{a}R^{b}_{xy}\,\hat{b}_y + {}^{a}R^{b}_{xz}\,\hat{b}_z\right) \\
+ \; & v_y \left({}^{a}R^{b}_{yx}\,\hat{b}_x + {}^{a}R^{b}_{yy}\,\hat{b}_y + {}^{a}R^{b}_{yz}\,\hat{b}_z\right) \\
+ \; & v_z \left({}^{a}R^{b}_{zx}\,\hat{b}_x + {}^{a}R^{b}_{zy}\,\hat{b}_y + {}^{a}R^{b}_{zz}\,\hat{b}_z\right)
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\vec{v} = \; & \left({}^{a}R^{b}_{xx}\,v_x + {}^{a}R^{b}_{yx}\,v_y + {}^{a}R^{b}_{zx}\,v_z\right)\hat{b}_x \\
+ \; & \left({}^{a}R^{b}_{xy}\,v_x + {}^{a}R^{b}_{yy}\,v_y + {}^{a}R^{b}_{zy}\,v_z\right)\hat{b}_y \\
+ \; & \left({}^{a}R^{b}_{xz}\,v_x + {}^{a}R^{b}_{yz}\,v_y + {}^{a}R^{b}_{zz}\,v_z\right)\hat{b}_z
\end{aligned}$$

Combining the previous expression for $\vec{v}$ with $\vec{v} = \bar{v}_x\,\hat{b}_x + \bar{v}_y\,\hat{b}_y + \bar{v}_z\,\hat{b}_z$, subsequent use of the transpose relationship in equation (2), and application of the definition of the elements of the ${}^{b}R^{a}$ rotation matrix from equations (1) and (3) proves equation (6).

$$\begin{bmatrix} \bar{v}_x \\ \bar{v}_y \\ \bar{v}_z \end{bmatrix} = \begin{bmatrix} {}^{a}R^{b}_{xx} & {}^{a}R^{b}_{yx} & {}^{a}R^{b}_{zx} \\ {}^{a}R^{b}_{xy} & {}^{a}R^{b}_{yy} & {}^{a}R^{b}_{zy} \\ {}^{a}R^{b}_{xz} & {}^{a}R^{b}_{yz} & {}^{a}R^{b}_{zz} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \underset{(2)}{=} \begin{bmatrix} {}^{b}R^{a}_{xx} & {}^{b}R^{a}_{xy} & {}^{b}R^{a}_{xz} \\ {}^{b}R^{a}_{yx} & {}^{b}R^{a}_{yy} & {}^{b}R^{a}_{yz} \\ {}^{b}R^{a}_{zx} & {}^{b}R^{a}_{zy} & {}^{b}R^{a}_{zz} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \underset{(1,3)}{=} {}^{b}R^{a} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$