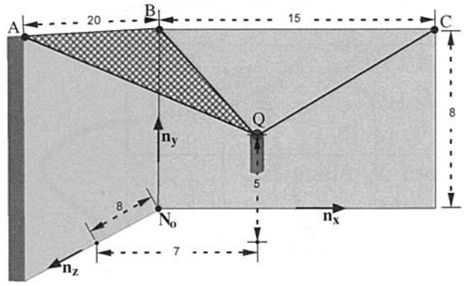
3.4 Example: Microphone cable surface area and normal (orthogonal walls)

A microphone Q is attached to three pegs A, B, and C by three cables. All three peg locations and the microphone location from point N_o are known. The surface area $|\vec{\Delta}|$ of the triangle formed by points A, B, and Q and a unit vector $\hat{u}$ perpendicular to the surface area are to be determined.



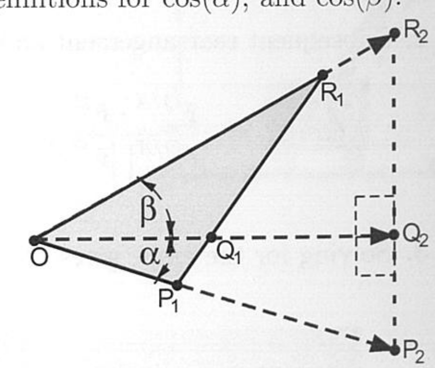
Quantity	Value
Distance from A to B	20 m
Distance from B to C	15 m
Distance from N_o to B	8 m
Q's horizontal measure from N_o along $\overline{BC}$	7 m
Q's vertical measure from N_o i.e., Qś height above N_o	5 m
Q's horizontal measure from N_o along $\overline{BA}$	8 m

Step-by-step solution to find $|\vec{\Delta}|$ and $\hat{u}$:

- Form Q's position vector from N_o (inspection): $\vec{r}^{\,Q/N_o} = 7\,\hat{n}_x + 5\,\hat{n}_y + 8\,\hat{n}_z$
  Form B's position vector from N_o (inspection): $\vec{r}^{\,B/N_o} = 8\,\hat{n}_y$
  Form B's position vector from A (inspection): $\vec{r}^{\,A/B} = 20\,\hat{n}_z$
- Form Q's position vector from B (vector addition and rearrangement):
  $$\vec{r}^{\,Q/B} = \vec{r}^{\,Q/N_o} - \vec{r}^{\,B/N_o} = 7\,\hat{n}_x + \text{-}3\,\hat{n}_y + 8\,\hat{n}_z$$
- The "*vector area*" is $\vec{\Delta} \triangleq \frac{1}{2}\,\vec{r}^{\,A/B} \times \vec{r}^{\,Q/B} = \frac{1}{2}\,(20\,\hat{n}_z) \times (7\,\hat{n}_x + \text{-}3\,\hat{n}_y + 8\,\hat{n}_z) = 30\,\hat{n}_x + 70\,\hat{n}_y$
- The magnitude of $\vec{\Delta}$ is the area, i.e., $|\vec{\Delta}| = \sqrt{30^2 + 70^2} = 76.16$
- The unit normal $\hat{u}$ in direction of $\vec{\Delta}$ is $\hat{u} = \dfrac{\vec{\Delta}}{|\vec{\Delta}|} = 0.394\,\hat{n}_x + 0.919\,\hat{n}_y$

3.5 Proof of sine addition formula with vector cross-products

The following figure shows a generic triangle $\Delta_{OP_1Q_1}$ that has one of its angles divided into two angles, namely α and β. Two right-triangles, namely $\Delta_{OP_2Q_2}$ and $\Delta_{OQ_2R_2}$, have been constructed as a geometrical starting point for the proof that follows and as a means to provide definitions for $\cos(\alpha)$, and $\cos(\beta)$.
Note: Starting construction courtesy of Dr. Alex Perkins.



The areas of triangles $\Delta_{OP_2R_2}$, $\Delta_{OP_2Q_2}$ and $\Delta_{OQ_2R_2}$ are

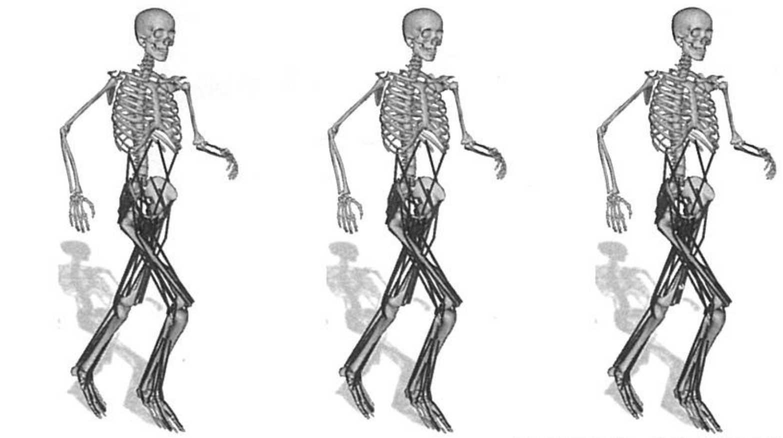
$$\text{Area}\,\Delta_{OP_2R_2} = \frac{1}{2}\,|\vec{OP}_2 \times \vec{OR}_2| = \frac{1}{2}\,|\vec{OP}_2|\,|\vec{OR}_2|\,\sin(\alpha+\beta)$$
$$\text{Area}\,\Delta_{OP_2Q_2} = \frac{1}{2}\,|\vec{OP}_2 \times \vec{OQ}_2| = \frac{1}{2}\,|\vec{OP}_2|\,|\vec{OQ}_2|\,\sin(\alpha)$$
$$\text{Area}\,\Delta_{OQ_2R_2} = \frac{1}{2}\,|\vec{OQ}_2 \times \vec{OR}_2| = \frac{1}{2}\,|\vec{OQ}_2|\,|\vec{OR}_2|\,\sin(\beta)$$

Using $\text{Area}\,\Delta_{OP_2R_2} = \text{Area}\,\Delta_{OP_2Q_2} + \text{Area}\,\Delta_{OQ_2R_2}$ and the definitions of $\cos(\alpha)$ and $\cos(\beta)$ gives

$$|\vec{OP}_2|\,|\vec{OR}_2|\,\sin(\alpha+\beta) = |\vec{OP}_2|\,|\vec{OQ}_2|\,\sin(\alpha) + |\vec{OQ}_2|\,|\vec{OR}_2|\,\sin(\beta)$$
$$\sin(\alpha+\beta) = \frac{|\vec{OQ}_2|}{|\vec{OR}_2|}\,\sin(\alpha) + \frac{|\vec{OQ}_2|}{|\vec{OP}_2|}\,\sin(\beta)$$
$$\sin(\alpha+\beta) = \cos(\beta)\,\sin(\alpha) + \cos(\alpha)\,\sin(\beta)$$

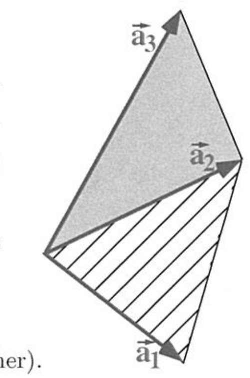
Chapter 4

Vector basis



Courtesy Dr. Sam Hamner

Why use a vector basis? (see examples in Hw 1, 2, 3)



Unit vectors are sign-posts, e.g., up, down, left, right, etc. A ***vector basis*** consisting of three orthogonal unit vectors provide a way to "give directions" in 3D space. Conventions for specifying unit vectors depend on the analyst and field of study, e.g., biomechanics, aeronautics, vehicle dynamics, statics, etc.*

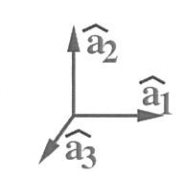
The vectors $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ shown right form a three-dimensional vector basis. Notice the basis is ***right-handed***,[a] but is not an ***orthogonal basis***[b] or ***unitary basis***.[c]

[a] The basis is ***right-handed*** (or ***dextral***) because $\vec{a}_1 \times \vec{a}_2 \cdot \vec{a}_3 > 0$.
[b] An ***orthogonal basis*** has ***mutually perpendicular*** (orthogonal) basis vectors (90° to each other).
[c] A ***unitary basis*** has unit basis vectors.

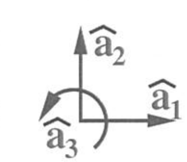
One	vector basis is useful for simple directions	(e.g., point Q from point O via $\vec{r}^{\,Q/O}$ – see Chapter 3).
Two	vector bases are useful for relative orientation	(e.g., aircraft A in Earth E via ${}^{A}R^{E}$ – see Chapter 5).
Multiple	vector bases are useful for multibody force and motion analyses.	

* For example, a vector basis for Earth's surface is ***NED*** (locally **N**orth/**E**ast/**D**own). A basis that orients Earth relative to other celestial objects is ***ECEF*** (**E**arth-**C**entered/**E**arth-**F**ixed) with a unit vector pointing from Earth's center to 0 longitude and 0 latitude, a second unit vector pointing to geometric North, and a third unit vector perpendicular to the other two.

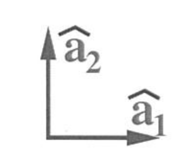
4.1 What is a vector basis?



A ***vector basis*** is a set of linearly independent vectors that span a space (e.g., the 3D space in which we live). Each linearly independent vector is called a ***basis vector*** for the space.

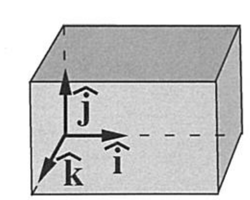


It is conventional to use a ***right-handed basis*** and common to use a ***orthogonal unitary basis***.[a] A 3D right-handed orthogonal unitary basis has various visual representations (shown to the right). Note: When $\hat{a}_3$ is absent, it is implied by the ***right-hand rule***.

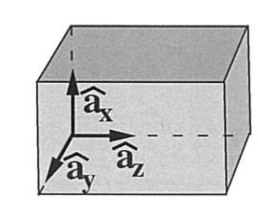


Note: A set of three vectors with an intrinsic order, e.g., $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, is called ***right-handed*** when $\vec{a}_1 \times \vec{a}_2 \cdot \vec{a}_3 > 0$. Alternately, the set is ***left-handed*** when $\vec{a}_1 \times \vec{a}_2 \cdot \vec{a}_3 < 0$. The orthogonal unit vectors $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ are ***right-handed*** when $\hat{a}_x \times \hat{a}_y = \hat{a}_z$.

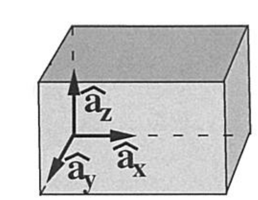
[a] To physically demonstrate an orthogonal vector basis, hold your right hand with the thumb, forefinger, and middle finger pointing in orthogonal directions. Chapter 5 deals with ***rotation matrix*** and is summarized with two hands (each with a vector basis) and the question "how do I relate two vector bases"
.



Right-handed basis

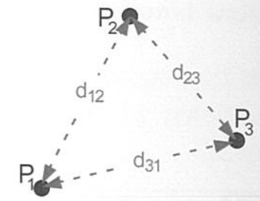


Right-handed basis



Left-handed basis

4.2 Rigid and non-rigid bases



When the magnitude of each basis vector in a vector basis is **constant** and the angles between bases vectors are **constant**, the basis is a "***rigid vector basis***".

When the magnitude of a basis vector in a vector basis is **variable** or an angle between two basis vectors are **variable**, the basis is a "***non-rigid vector basis***".

For example, the previous figure shows distinct non-collinear points P_1, P_2, P_3 and the non-zero distances between them d_{12}, d_{23}, d_{31}. One way to construct a basis is from the vector $\vec{a}_1$ directed from P_1 to P_2, the vector $\vec{a}_2$ directed from P_1 to P_3, and $\vec{a}_3 = \vec{a}_1 \times \vec{a}_2$. This is a ***rigid vector basis*** if all the distances are constant whereas it is a ***non-rigid vector basis*** if d_{12} is variable.

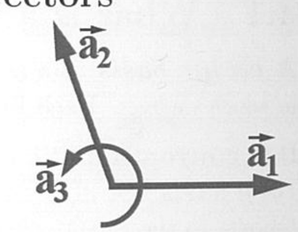
Note: Even though a ***rigid vector basis*** can sometimes be associated with a unique ***reference frame***, a reference frame contains an infinite number of vector bases. For example, a rigid basis consisting of $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ may be fixed in a reference frame A. However, one is free to fix other rigid bases (e.g., $\widehat{a}_x$, $\widehat{a}_y$, $\widehat{a}_z$) in A.

Note: A ***reference frame*** is a *rigid object* that can be constructed with as few as three non-collinear points whose distance from each other are constant. Reference frames are discussed in Chapter 7 and differ from a rigid basis in that at least one point must be fixed in a reference frame whereas a rigid basis is not associated with a point. Differences between reference frames and rigid basis are most relevant when dealing with **translational** kinematics (e.g., velocity and acceleration).

4.3 Non-orthogonal basis

There are situations in which it is sensible to use a ***non-orthogonal basis***. For example, a non-orthogonal basis plays an important role is in determining the volume, centroid, and inertia properties of a tetrahedron, a shape used by CAD/CAE programs for constructing nearly any geometrical object.
Non-orthogonal bases are also useful in motion studies (e.g., ***gait studies***) involving irregularly-shaped objects (e.g., human bones) that require ***markers*** (devices which track the location of a single point) on easily-identifiable, physically-meaningful locations (e.g., ***anatomic landmarks***). It is easier (and more physically meaningful) to construct a non-orthogonal basis out of basis vectors which are aligned with markers of interest (e.g., pointing from one marker to another marker).

4.4 Creating various 3D bases from two non-parallel vectors



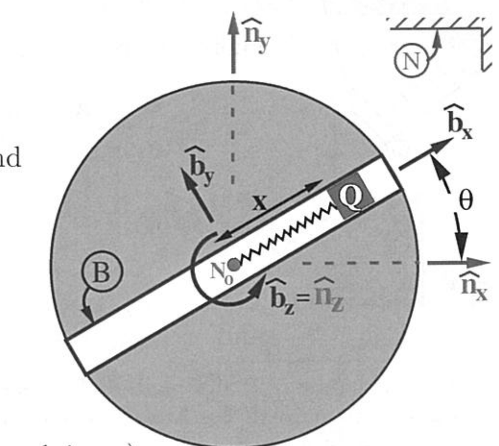
One way to construct a 3D vector basis from two non-parallel vectors $\vec{a}_1$ and $\vec{a}_2$ is with: $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3 \triangleq \vec{a}_1 \times \vec{a}_2$.

One way to construct a 3D right-handed ***orthogonal basis*** from $\vec{a}_1$ and $\vec{a}_2$ is with: $\vec{a}_1$, $\vec{a}_3 \times \vec{a}_1$, and $\vec{a}_3$.

4.5 Concept: What is the vector vs. how is it expressed

The following figure shows a particle Q sliding along a straight track B. Track B spins in a reference frame N. A spring with linear spring constant k connects Q to point N_o (N_o is fixed in both B and N).
Vector bases $\widehat{n}_x$, $\widehat{n}_y$, $\widehat{n}_z$ and $\widehat{b}_x$, $\widehat{b}_y$, $\widehat{b}_z$ are fixed in N and B, respectively with:

- $\widehat{n}_x$ directed horizontally right
- $\widehat{n}_z = \widehat{b}_z$ parallel to B's axis of rotation in N
- $\widehat{b}_x$ directed along the track from N_o to Q



Using the geometry of a right triangle and definitions of sine and cosine, unit vectors $\widehat{n}_x$ and $\widehat{n}_y$ are related to $\widehat{b}_x$ and $\widehat{b}_y$ as

$$\widehat{b}_x = \cos(\theta)\,\widehat{n}_x + \sin(\theta)\,\widehat{n}_y$$
$$\widehat{b}_y = -\sin(\theta)\,\widehat{n}_x + \cos(\theta)\,\widehat{n}_y$$

The point of this example is to clarify two distinct concepts:
- What is the vector (the name to the left of the equal sign =)
- How is the vector expressed (the expression on the right of the equal sign =)

$\vec{r}$ (Q's position vector from N_o) can be ***expressed*** in various bases.

Expressed in terms of $\widehat{b}_x$, $\widehat{b}_y$, $\widehat{b}_z$. $\vec{r} = x\,\widehat{b}_x$

Expressed in terms of $\widehat{n}_x$, $\widehat{n}_y$, $\widehat{n}_z$. $\vec{r} = x\,[\cos(\theta)\,\widehat{n}_x + \sin(\theta)\,\widehat{n}_y]$

$\vec{F}$ (the spring force on Q) can be ***expressed*** in various bases.

Expressed in terms of $\widehat{b}_x$, $\widehat{b}_y$, $\widehat{b}_z$. $\vec{F} = -k\,x\,\widehat{b}_x$

Expressed in terms of $\widehat{n}_x$, $\widehat{n}_y$, $\widehat{n}_z$. $\vec{F} = -k\,x\,[\cos(\theta)\,\widehat{n}_x + \sin(\theta)\,\widehat{n}_y]$

${}^{B}\vec{v}^{Q}$ (Q's velocity in B) can be ***expressed*** in various bases.

Expressed in terms of $\widehat{b}_x$, $\widehat{b}_y$, $\widehat{b}_z$. ${}^{B}\vec{v}^{Q} = \dot{x}\,\widehat{b}_x$

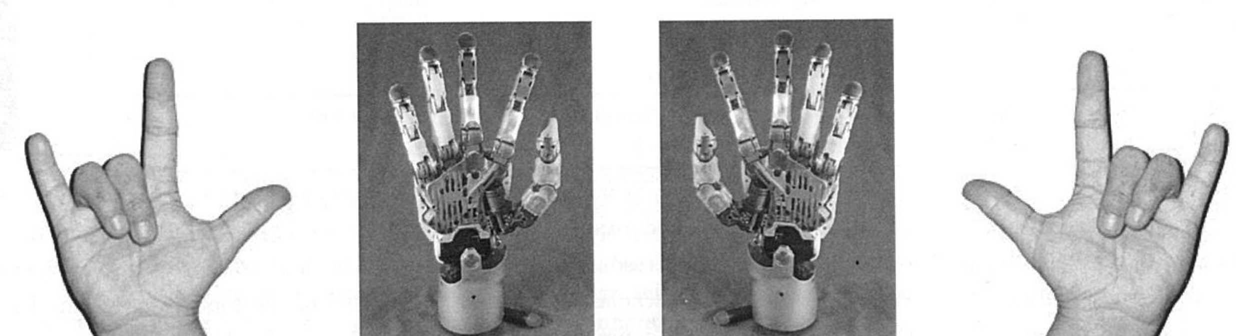
Expressed in terms of $\widehat{n}_x$, $\widehat{n}_y$, $\widehat{n}_z$. ${}^{B}\vec{v}^{Q} = \dot{x}\,[\cos(\theta)\,\widehat{n}_x + \sin(\theta)\,\widehat{n}_y]$

${}^{N}\vec{v}^{Q}$ (Q's velocity in N) can be ***expressed*** in various bases. Note: ${}^{N}\vec{v}^{Q} \neq {}^{B}\vec{v}^{Q}$. These are different vectors.

Expressed in terms of $\widehat{b}_x$, $\widehat{b}_y$, $\widehat{b}_z$. ${}^{N}\vec{v}^{Q} = \dot{x}\,\widehat{b}_x + x\,\dot{\theta}\,\widehat{b}_y$

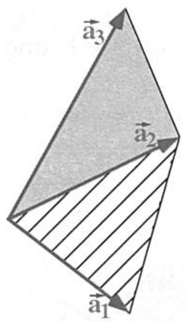
Expressed in terms of $\widehat{n}_x$, $\widehat{n}_y$, $\widehat{n}_z$. ${}^{N}\vec{v}^{Q} = [\dot{x}\,\cos(\theta) - x\,\dot{\theta}\,\sin(\theta)]\,\widehat{n}_x + [\dot{x}\,\sin(\theta) + x\,\dot{\theta}\,\cos(\theta)]\,\widehat{n}_y$

The important point to remember is: A vector is not changed by ***expressing*** it in a different basis.



Pictures and renderings of DARPA's Revolutionizing Prosthetic hand.
Courtesy of HDT Engineering Services, Inc. and Kinea Design, LLC.

4.6 Expressing a vector

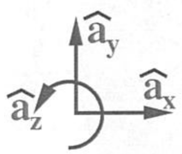


Given an arbitrary vector $\vec{v}$ and a set of three non-coplanar (but not necessarily orthogonal or unit) vectors $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, one can *express* $\vec{v}$ in terms of $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ as

$$\vec{v} = v_1\,\vec{a}_1 + v_2\,\vec{a}_2 + v_3\,\vec{a}_3 \tag{1}$$

where v_1, v_2, v_3 are scalars quantities (e.g., numbers or functions of time t) equal to[†]

$$v_1 = \frac{\vec{v} \cdot (\vec{a}_2 \times \vec{a}_3)}{\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)} \qquad v_2 = \frac{\vec{v} \cdot (\vec{a}_3 \times \vec{a}_1)}{\vec{a}_2 \cdot (\vec{a}_3 \times \vec{a}_1)} \qquad v_3 = \frac{\vec{v} \cdot (\vec{a}_1 \times \vec{a}_2)}{\vec{a}_3 \cdot (\vec{a}_1 \times \vec{a}_2)} \tag{2}$$



When $\widehat{a}_x$, $\widehat{a}_y$, $\widehat{a}_z$ are orthogonal unit vectors, equation (2) simplifies to

$$v_1 = \vec{v} \cdot \widehat{a}_x \qquad v_2 = \vec{v} \cdot \widehat{a}_y \qquad v_3 = \vec{v} \cdot \widehat{a}_z \tag{3}$$

and v_1, v_2, v_3 are called the $\widehat{a}_x$, $\widehat{a}_y$, $\widehat{a}_z$ *measures* of $\vec{v}$.[a]

[†] To prove equation (2), dot multiply both sides of equation (1) with $\vec{a}_2 \times \vec{a}_3$ to get $\vec{v} \cdot (\vec{a}_2 \times \vec{a}_3) \underset{(1)}{=} v_1\,\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)$. Isolate v_1 to arrive at the first expression in equation (2). Proceed similarly to find v_2 and v_3.
Note: Since $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ are non-parallel, non-coplanar vectors $\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) \neq 0$.

[a] *Measures* of $\vec{v}$ are scalars whereas *components* of $\vec{v}$ are vectors.
For example, the $\widehat{a}_x$ *measure* of $\vec{v}$ is v_1 whereas the $\widehat{a}_x$ *component* of $\vec{v}$ is $(\vec{v} \cdot \widehat{a}_x)\,\widehat{a}_x = v_1\,\widehat{a}_x$.

4.7 Expressing a vector basis in terms of another vector basis

One set of basis vectors (e.g., $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$) can be expressed in terms of another set of basis vectors (e.g., $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$) with the scalar functions R_{ij} $(i,j = 1,2,3)$ as either

$$\begin{aligned} \vec{b}_1 &= R_{11}\,\vec{a}_1 + R_{12}\,\vec{a}_2 + R_{13}\,\vec{a}_3 \\ \vec{b}_2 &= R_{21}\,\vec{a}_1 + R_{22}\,\vec{a}_2 + R_{23}\,\vec{a}_3 \\ \vec{b}_3 &= R_{31}\,\vec{a}_1 + R_{32}\,\vec{a}_2 + R_{33}\,\vec{a}_3 \end{aligned} \quad \text{or} \quad \begin{bmatrix} \vec{b}_1 \\ \vec{b}_1 \\ \vec{b}_1 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_1 \\ \vec{a}_1 \end{bmatrix}$$

When $\vec{a}_i$ and $\vec{b}_i$ $(i = 1,2,3)$ are both right-handed, orthogonal, unitary bases, the matrix relating $\vec{b}_i$ to $\vec{a}_i$ is called the ${}^bR^a$ ***rotation matrix*** and has many special properties as described in Chapter 5.

4.8 Optional**: The language, history, and culture of "left" and "right"

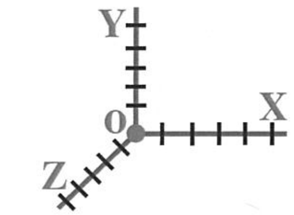
Language	Word	Translation	Meaning	More info
English	right	right	correct, "you are right"	Engineers like being "right"
English	left	left	"left out"	
French	right	droit	adroit means to the right or skillful	
French	left	gauche	socially clumsy	http://www.gauche.com
Latin	right	dexter	nimble, dexterous	
Latin	left	sinistre	dark and mysterious	
Greek	right	orthos	root of the word orthogonal	Also dexion
Greek	left	skalos	awkward, ill-omened	

Left-handers include Alexander the Great, Julius Caesar, Leonardo, Michelangelo, Raphael, Newton, Curie, Henry Ford, Napoleon, and a disproportion of Nobel-prize winners and recent U.S. presidents, including Gerald Ford, Ronald Regan (ambidextrous), Bill Clinton, and Barack Obama. One ultrasound study showed 90% of in-utero babies sucking their right thumb. In a study of 100,000 students taking the SAT (Scholastic Aptitude Test), 20% of the top-scoring group was left-handed, twice the 10% rate of left-handed students in the general population. In 2007, gene LRRTM1 was associated with both left-handedness and an increased chance of schizophrenia.

4.9 Optional**: Coordinate system vs. vector bases

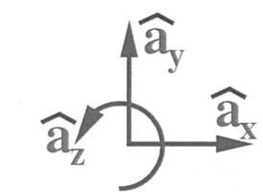
A ***coordinate system*** is a set of scalar quantities, typically angles or distances, used in specifying the location of points, curves, surfaces, and solids. A ***coordinate*** is a single scalar in the set.[1] A ***generalized coordinate*** is a scalar quantity that is useful in locating points, curves, surfaces, and solids but is not necessarily associated with a coordinate system such as a Cartesian, cylindrical, or spherical[2] coordinate system.[3] In other words a generalized coordinate is a more general type of coordinate. Generalized coordinates play an increasingly important role in geometry and motion.

Coordinate system	Method for locating points
Cartesian coordinate system	3 distances measures, e.g., (x, y, z)
Cylindrical coordinate system	2 distances and 1 angle, e.g., (r, θ, z)
Spherical coordinate system	1 distance and 2 angles, e.g., (ρ, θ, ϕ)



Cartesian coordinate system

The most famous coordinate system is a rectangular ***Cartesian coordinate system*** which consists of three mutually-perpendicular lines, called ***coordinate axes***, along which measurements are done and which all intersect at one point called the ***origin***. The differences between a Cartesian coordinate system and a vector basis are highlighted below.[a]



Vector basis

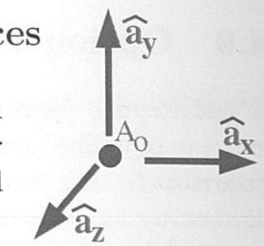
[a] It is usually more efficient to use generalized coordinates and vector bases than coordinate systems.

- A Cartesian coordinate system has an **origin** and a **set of coordinates**. A basis does not.
- A Cartesian coordinate system has **coordinate axes** along which measurements are done. A basis does not.
- A Cartesian coordinate system does not intrinsically have a basis - although one can easily be constructed by introducing unit vectors that are oriented parallel to the coordinate axes and whose sense is determined by the positive direction along the coordinate axes.

Note: A ***vector basis*** is defined as a linearly independent set of vectors that "***span a space***".
A set of vectors $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ is said to "***span 3D space***" (e.g., Earth's 3D space) if and only if any arbitrary vector $\vec{v}$ in 3D space can be written as a "***linear combination***" of $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$. A ***linear combination*** of vectors $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ is defined as $\sum_{i=1}^{n} v_i\,\vec{a}_i$ where v_i are scalar quantities. As shown in Section 4.6, if $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ form a 3D vector basis, any arbitrary 3D vector can be written $\vec{v} = v_1\,\vec{a}_1 + v_2\,\vec{a}_2 + v_3\,\vec{a}_3$. One way to form a basis is by "guess and check". As shown in Section 4.6, one may guess $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ form a 3D basis. The solution for v_1, v_2, v_3 checks if and only if $\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) \neq 0$, which means $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ are non-coplanar vectors.

[1] A coordinate may be a variable, constant, or specified function of time.
[2] A ***spherical coordinate system*** is useful for describing the location of a point on a sphere. When studying the motion of particles moving on the Earth, e.g., the geology of a particle in a river, it is helpful to use a spherical coordinate system because ρ is a constant and the number of variables in the analysis is decreased from three to two. Using a Cartesian coordinate system to study a particle moving on a sphere introduces an inherent relationship between x, y, and z, i.e., $x^2 + y^2 + z^2 =$ constant. Similarly, using a polar coordinate system necessitates an inherent relationship between r, θ, and z, i.e., $r^2 + z^2 =$ constant. However, Homework 5.22 shows that spherical coordinates have an inherent singularity at the "North" and "South" pole.
[3] Other related coordinates include curvilinear, Plucker, canonical, intrinsic, parallel, elliptic, ellipsoidal, prolate spheroidal, oblate spheroidal, conical, parabolic, paraboloidal, toroidal, bispherical, biangular, etc.

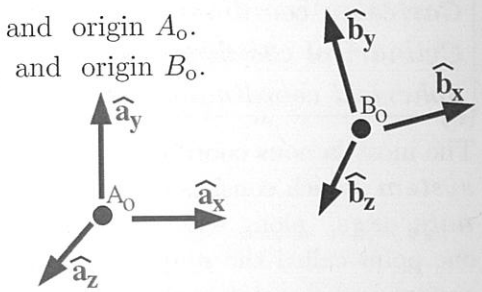
4.10 Optional**: Rigid frame and transformation matrices



A ***rigid frame*** is the combination of a ***rigid vector basis*** and an ***origin*** point. A rigid frame is a useful measuring device for kinematics and multibody dynamics. For example, the figure to the right shows a rigid frame **A** having right-handed orthogonal unit vectors $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and an origin A_o.

Optional**: Transformation matrices (for robotics, graphics, ...)

The following figure shows two ***rigid frames*** A and B.
Rigid frame A has right-handed orthogonal unit vectors $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and origin A_o.
Rigid frame B has right-handed orthogonal unit vectors $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ and origin B_o.

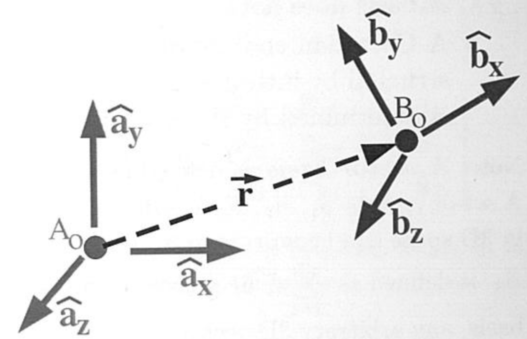


The 4×4 ***transformation matrix*** ${}^AT^B$ stores the orientation of A and B in a 3×3 ${}^aR^b$ ***rotation matrix*** (described in Chapter 5) and stores the $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ measures of B_o's position vector from A_o in a 3×1 sub-matrix as shown below.

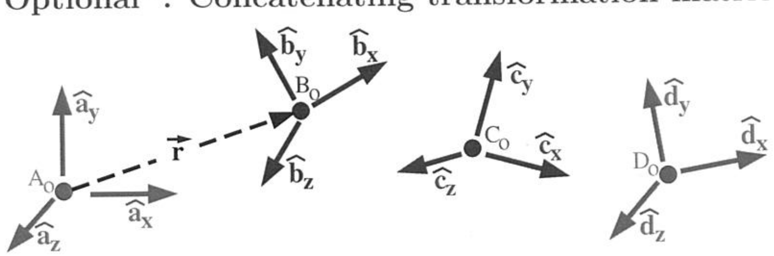
$$ {}^AT^B \triangleq \left[\begin{array}{ccc|c} & {}^aR^b & & \left[\vec{r}^{\,B_o/A_o}\right]_a \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} \hat{a}_x \cdot \hat{b}_x & \hat{a}_x \cdot \hat{b}_y & \hat{a}_x \cdot \hat{b}_z & \hat{a}_x \cdot \vec{r}^{\,B_o/A_o} \\ \hat{a}_y \cdot \hat{b}_x & \hat{a}_y \cdot \hat{b}_y & \hat{a}_y \cdot \hat{b}_z & \hat{a}_y \cdot \vec{r}^{\,B_o/A_o} \\ \hat{a}_z \cdot \hat{b}_x & \hat{a}_z \cdot \hat{b}_y & \hat{a}_z \cdot \hat{b}_z & \hat{a}_z \cdot \vec{r}^{\,B_o/A_o} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \quad (4) $$

Numerical values representative of the schematic to the right can be stored in a 4×4 ***transformation table***, e.g., as

${}^AT^B$	$\hat{b}_x$	$\hat{b}_y$	$\hat{b}_z$	$\vec{r}^{\,B_o/A_o}$
$\hat{a}_x$	0.6958	-0.7165	0.0500	4.2
$\hat{a}_y$	0.7103	0.6968	0.0100	3.3
$\hat{a}_z$	-0.1063	-0.0339	0.9938	-1.2
	0	0	0	1



Optional**: Concatenating transformation matrices for rigid frames A, B, C, D



The 4×4 ***transformation matrix*** ${}^AT^D$ relates the orientation and position of rigid frames A and D and can be computed by matrix multiplication as shown below.

$$ {}^AT^D = {}^AT^B * {}^BT^C * {}^CT^D $$

Optional**: Inverse of a transformation matrix

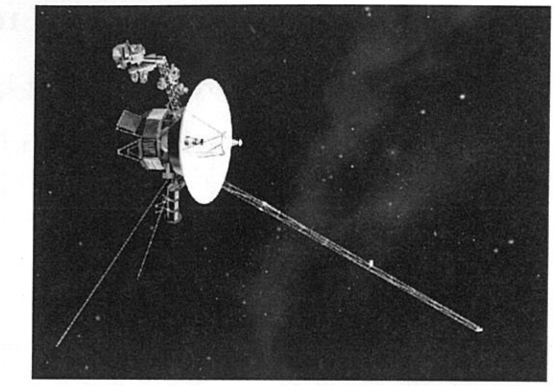
Since rotation matrices are orthogonal, ${}^bR^a$ has the special property ${}^bR^a = ({}^aR^b)^{-1} = ({}^aR^b)^T$.

Using this special rotation matrix property, the inverse of a transformation matrix can be efficiently calculated as shown right (more efficient and accurate than inverting a 4×4 matrix).

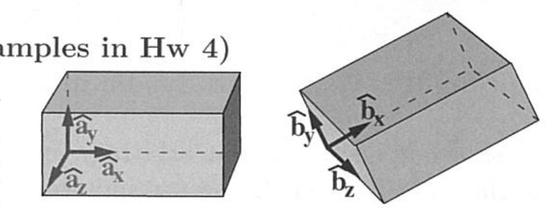
$$ {}^BT^A = \left({}^AT^B\right)^{-1} = \left[\begin{array}{ccc|c} & {}^bR^a & & -{}^bR^a * \left[\vec{r}^{\,B_o/A_o}\right]_a \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

Chapter 5

Rotation matrices I



Rotation matrices and vector bases (see examples in Hw 4)



The figure to the right shows two ***right-handed, unitary orthogonal bases*** a and b whose basis vectors are $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$, respectively. ***Orientation*** between a and b can be described with a 3 × 3 ***rotation matrix*** ${}^aR^b$, whose elements ${}^aR^b_{ij}$ $(i,j = x,y,z)$ are defined in equation (1).

$$ {}^aR^b_{ij} \triangleq \hat{a}_i \cdot \hat{b}_j \underset{(2.2)}{=} \cos\angle(\hat{a}_i, \hat{b}_j) \qquad (i,j = x,y,z) \quad (1) $$

The definition of dot-product in equation (2.2) shows ${}^aR^b_{ij}$ is equal to the cosine of the angle between $\hat{a}_i$ and $\hat{b}_j$.[a]

[a] A rotation matrix is also called a ***direction cosine matrix*** whose elements are ***direction cosines***. Section 5.4.1 gives an example of calculating angles from a rotation matrix. Chapter 8 shows the orientation of a in b can be characterized in various ways, including Euler angles, Euler parameters (quaternions), and Rodrigues parameters.

A ***rotation matrix*** R is an orthogonal matrix, which means the transpose of R is equal to the inverse of R, i.e., $R^T = R^{-1}$. Section 5.4.2 gives an example of calculating the inverse of a rotation matrix.

$$ {}^bR^a = ({}^aR^b)^{-1} = ({}^aR^b)^T \quad (2) $$

Since rotation matrices are orthogonal, it is convenient to store the orientation information in a ***rotation table*** that can be read horizontally or vertically.[a]

${}^aR^b$	$\hat{b}_x$	$\hat{b}_y$	$\hat{b}_z$
$\hat{a}_x$	$\hat{a}_x \cdot \hat{b}_x$	$\hat{a}_x \cdot \hat{b}_y$	$\hat{a}_x \cdot \hat{b}_z$
$\hat{a}_y$	$\hat{a}_y \cdot \hat{b}_x$	$\hat{a}_y \cdot \hat{b}_y$	$\hat{a}_y \cdot \hat{b}_z$
$\hat{a}_z$	$\hat{a}_z \cdot \hat{b}_x$	$\hat{a}_z \cdot \hat{b}_y$	$\hat{a}_z \cdot \hat{b}_z$

(3)

[a] Section 5.7.1 gives the proof of equation (2). If one or both of the bases are non-orthogonal, a rotation matrix (not a rotation table) is used because the inverse of a non-orthogonal matrix is not its transpose.

The rotation matrix ${}^aR^d$ can be formed by successive matrix multiplication of the ${}^aR^b$, ${}^bR^c$, and ${}^cR^d$ rotation matrices. Section 5.5.4 gives an example of successive matrix multiplication.

$$ {}^aR^d = {}^aR^b * {}^bR^c * {}^cR^d \quad (4) $$

5.1 Uses for the rotation matrix ${}^aR^b$ (for geometry, statics, motion analysis, stress ...)

- Finding the **dot-product** between the unit vectors $\hat{a}_i$ and $\hat{b}_j$, $(i,j = x,y,z)$
- Calculating the **angle** between the unit vectors $\hat{a}_i$ and $\hat{b}_j$, $(i,j = x,y,z)$
- Calculating the **dot-product** or **cross-product** between a vector $\vec{v}$ and a vector $\vec{w}$, each of which may be expressed in terms of $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and/or $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ (e.g., see Section 5.4.3).
- **Expressing** a vector written in terms of $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ in terms of $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ (or vice-versa)
- Calculating other rotation matrices e.g., ${}^aR^d = {}^aR^b * {}^bR^c * {}^cR^d$ [see equation (6)].
- Relating the column matrix representation of a vector $\vec{v}$ expressed in a vector basis a to its column matrix representation in a vector basis b, e.g., $[\vec{v}]_a = {}^aR^b\,[\vec{v}]_b$