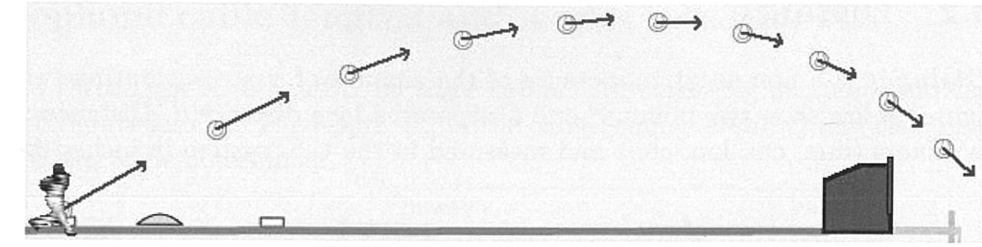
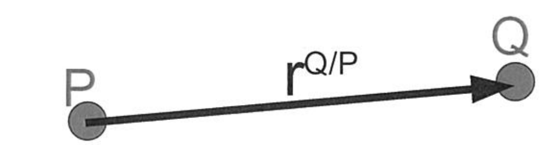
Chapter 3

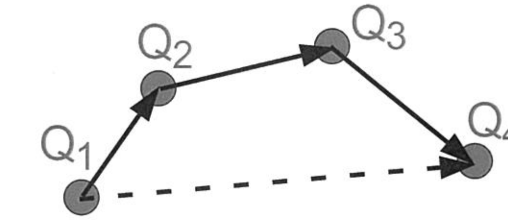


Position vectors and vector geometry

3.1 Position of a point (see examples in Hw 3)



A point is a location in space and has no spatial dimension (no height, width, or depth). A point's location can be measured with a ***position vector*** that characterizes its position from another point. The figure to the right shows two points P and Q. The symbol $\vec{r}^{\,Q/P}$ denotes Q's position vector from P.



Usually, a position vector is formed by ***inspection*** or ***vector addition***. Shown to the right are points Q_1, Q_2, Q_3, Q_4. The quantity $\vec{r}^{\,Q_4/Q_1}$ (Q_4's position vector from Q_1) is formed by vector addition as shown in equation (1).

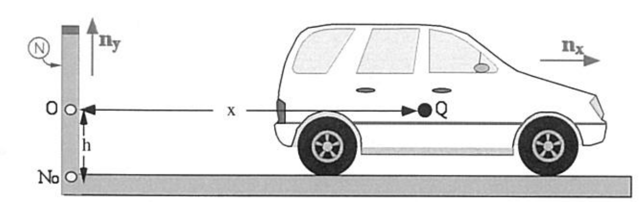
Position vectors are very useful for geometry.

$$\boxed{\vec{r}^{\,Q_4/Q_1} = \vec{r}^{\,Q_4/Q_3} + \vec{r}^{\,Q_3/Q_2} + \vec{r}^{\,Q_2/Q_1}} \qquad (1)$$

Note: Since a body contains an infinite number of points, a body does not have a position.

[a] Other geometry examples include ***angles*** [Homework 2.5 and Homework 4.16], ***distance*** [Homework 2.11], ***area*** [Homework 2.15], and ***location*** [Homework 2.20, Homework 4.17, and Homework 7.2].

Position vector example



$$\vec{r}^{\,O/N_o} = h\,\widehat{\mathbf{n}}_y \qquad \text{(inspection)}$$

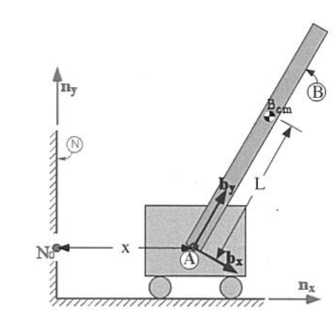
$$\vec{r}^{\,Q/O} = x\,\widehat{\mathbf{n}}_x \qquad \text{(inspection)}$$

$$\vec{r}^{\,Q/N_o} = \vec{r}^{\,Q/O} + \vec{r}^{\,O/N_o} = h\,\widehat{\mathbf{n}}_y + x\,\widehat{\mathbf{n}}_x$$

Position vector example (inverted pendulum on cart)

By inspection of the figure to the right, one can determine:

- $\vec{r}^{\,A/N_o} = x\,\widehat{\mathbf{n}}_x$ (Point A's position vector from point N_o)
- $\vec{r}^{\,B_{cm}/A} = L\,\widehat{\mathbf{b}}_y$ (Point B_{cm}'s position vector from point A)

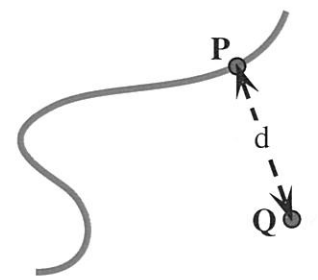


B_{cm}'s position vector from N_o is computed with vector addition, i.e., .

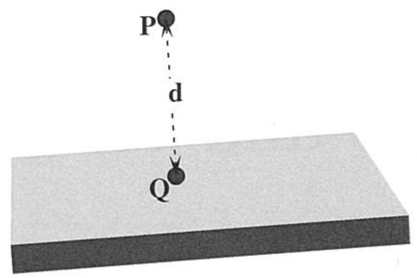
$$\vec{r}^{\,B_{cm}/N_o} = \vec{r}^{\,B_{cm}/A} + \vec{r}^{\,A/N_o} = L\,\widehat{\mathbf{b}}_y + x\,\widehat{\mathbf{n}}_x$$

3.2 Distance

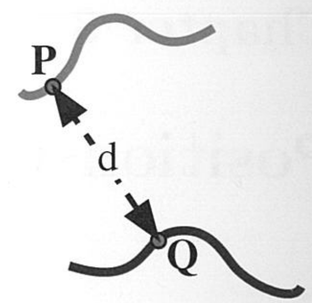
Distance is a non-negative measure of the amount of space separating two points.[1] [2] For example, the figures below show two points P and Q separated by a distance d. Distances are measured in the SI system in meters (mm, cm, km, etc.) and measured in the U.S. system in inches, feet, yards, and miles.



Distance between a point and a curve



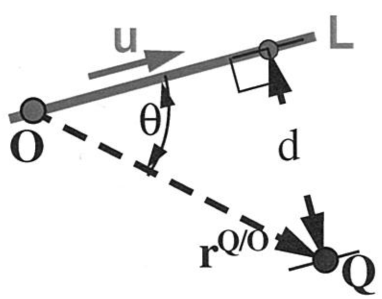
Distance between a point and a plane



Distance between two curves

One way to measure the distance d between a point P and a point Q is with the magnitude of Q's position vector from P.

$$\boxed{d \;=\; |\vec{r}^{\,Q/P}| \;=\; +\sqrt{\vec{r}^{\,Q/P} \cdot \vec{r}^{\,Q/P}}} \qquad (2)$$



Distance between a line and a point

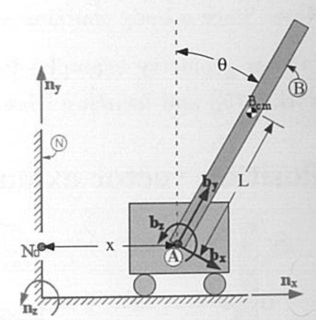
The figure to the left shows a line L that passes through a point O and is parallel to the unit vector $\hat{\mathbf{u}}$. The distance d between line L and a point Q can be calculated as follows.

$$\boxed{d \;=\; |\vec{r}^{\,Q/O} \times \hat{\mathbf{u}}| \;\underset{(6)}{=}\; |\vec{r}^{\,Q/O}| \,\sin(\theta)} \qquad (3)$$

Distance example (inverted pendulum on cart)

Referring to the figure on the right, the distance between N_o and B_{cm} is

$$\begin{aligned} d \;&=\; +\sqrt{\vec{r}^{\,B_{cm}/N_o} \cdot \vec{r}^{\,B_{cm}/N_o}} \\ &=\; +\sqrt{(x\,\hat{\mathbf{n}}_x \,+\, L\,\hat{\mathbf{b}}_y) \cdot (x\,\hat{\mathbf{n}}_x \,+\, L\,\hat{\mathbf{b}}_y)} \\ &=\; +\sqrt{x^2 \,+\, 2\,x\,L\,\sin(\theta) \,+\, L^2} \end{aligned}$$

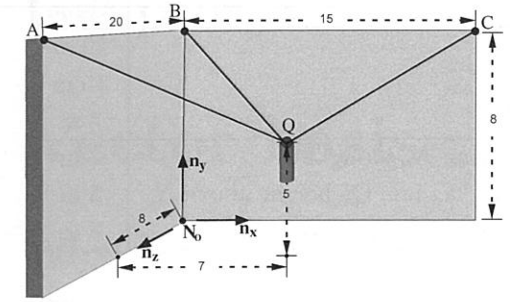


[1] Although distances are non-negative quantities, a ***measure*** may be negative or positive as it is associated with a *sense*.

[2] The ***distance between a point and a curve*** is defined as the distance from the point to the closest point on the curve. The ***distance between a point and a plane*** is the distance from the point to the closest point on the plane. The ***distance between two curves*** is defined to be the distance between the two closest points on the curves. ***Arc-length*** (distance along a curve) is defined as the limit of the sum of distances between sequential points on a curve (as the points get closer together).

3.3 Example: Microphone cable lengths and angles (orthogonal walls)

A microphone Q is attached to three pegs A, B, and C by three cables. All three peg locations and the microphone location from point N_o are known. The length L_A of the cable joining A and Q and the angle ϕ between line $\overline{AQ}$ and line $\overline{AB}$ are to be determined.



Quantity	Value
Distance from A to B	20 m
Distance from B to C	15 m
Distance from N_o to B	8 m
Q's measure from N_o along $\overline{BC}$	7 m
Q's height above N_o	5 m
Q's measure from N_o along $\overline{BA}$	8 m

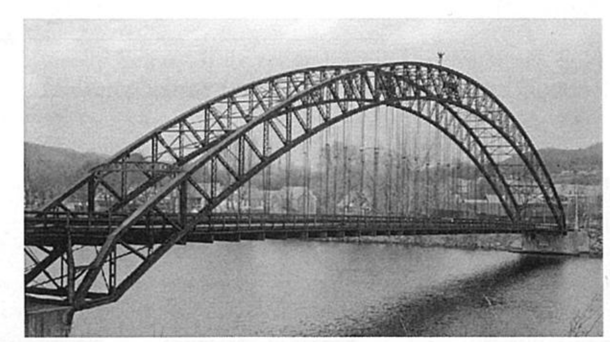
$$\vec{r}^{\,Q/N_o} \;=\; 7\,\hat{\mathbf{n}}_x \;+\; 5\,\hat{\mathbf{n}}_y \;+\; 8\,\hat{\mathbf{n}}_z$$

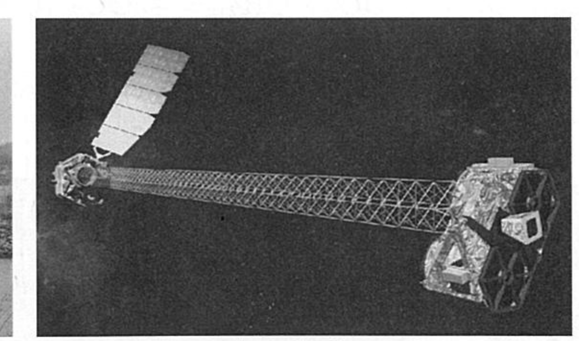
Step-by-step solution to find L_A (length of the cable joining A and Q):

1. Form A's position vector from N_o (inspection): $\vec{r}^{\,A/N_o} \;=\; 8\,\hat{\mathbf{n}}_y \;+\; 20\,\hat{\mathbf{n}}_z$
2. Form Q's position vector from A (vector addition and rearrangement):
$$\vec{r}^{\,Q/A} \;=\; \vec{r}^{\,Q/N_o} \;-\; \vec{r}^{\,A/N_o} \;=\; 7\,\hat{\mathbf{n}}_x \;+\; \text{-}3\,\hat{\mathbf{n}}_y \;+\; \text{-}12\,\hat{\mathbf{n}}_z$$
3. Calculate $\vec{r}^{\,Q/A} \cdot \vec{r}^{\,Q/A}$: $\quad (7\,\hat{\mathbf{n}}_x \,+\, \text{-}3\,\hat{\mathbf{n}}_y \,+\, \text{-}12\,\hat{\mathbf{n}}_z) \cdot (7\,\hat{\mathbf{n}}_x \,+\, \text{-}3\,\hat{\mathbf{n}}_y \,+\, \text{-}12\,\hat{\mathbf{n}}_z) \;=\; 202$
4. Form the magnitude of Q's position vector from A: $\quad L_A \;=\; \sqrt{\vec{r}^{\,Q/A} \cdot \vec{r}^{\,Q/A}} \;=\; \sqrt{202} \;=\; 14.2$

Step-by-step solution to find ϕ (angle between lines $\overline{AQ}$ and $\overline{AB}$):

1. The determination of the angle ϕ starts with the definition of the following dot-product
$$\vec{r}^{\,Q/A} \cdot \vec{r}^{\,B/A} \;\triangleq\; |\vec{r}^{\,Q/A}|\;|\vec{r}^{\,B/A}|\;\cos(\phi)$$
2. Subsequent rearrangement and substitution of the known quantities gives
$$\cos(\phi) \;=\; \frac{\vec{r}^{\,Q/A} \cdot \vec{r}^{\,B/A}}{|\vec{r}^{\,Q/A}|\;|\vec{r}^{\,B/A}|} \;=\; \frac{\vec{r}^{\,Q/A} \cdot \vec{r}^{\,B/A}}{20\;L_A} \;=\; \frac{(7\,\hat{\mathbf{n}}_x \,+\, \text{-}3\,\hat{\mathbf{n}}_y \,+\, \text{-}12\,\hat{\mathbf{n}}_z) \cdot (\text{-}20\,\hat{\mathbf{n}}_z)}{20 \,*\, 14.2} \;=\; \frac{240}{284}$$
3. Solving for the angle gives $\phi \;=\; \text{acos}\left(\frac{240}{284}\right) \;=\; 0.564 \text{ rad} \;=\; 32.32°$.

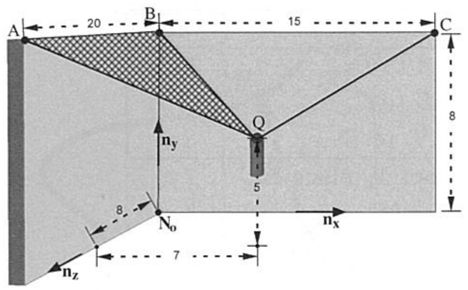




Vectors and geometry are heavily used for construction and structural design (e.g., bridges and space-structures)

3.4 Example: Microphone cable surface area and normal (orthogonal walls)

A microphone Q is attached to three pegs A, B, and C by three cables. All three peg locations and the microphone location from point N_o are known. The surface area $|\vec{\Delta}|$ of the triangle formed by points A, B, and Q and a unit vector $\hat{\mathbf{u}}$ perpendicular to the surface area are to be determined.



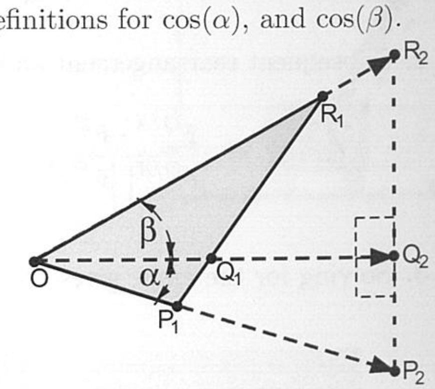
Quantity	Value
Distance from A to B	20 m
Distance from B to C	15 m
Distance from N_o to B	8 m
Q's horizontal measure from N_o along $\overline{BC}$	7 m
Q's vertical measure from N_o i.e., Qś height above N_o	5 m
Q's horizontal measure from N_o along $\overline{BA}$	8 m

Step-by-step solution to find $|\vec{\Delta}|$ and $\hat{\mathbf{u}}$:

- Form Q's position vector from N_o (inspection): $\vec{\mathbf{r}}^{Q/N_o} = 7\,\hat{\mathbf{n}}_x + 5\,\hat{\mathbf{n}}_y + 8\,\hat{\mathbf{n}}_z$
  Form B's position vector from N_o (inspection): $\vec{\mathbf{r}}^{B/N_o} = 8\,\hat{\mathbf{n}}_y$
  Form B's position vector from A (inspection): $\vec{\mathbf{r}}^{A/B} = 20\,\hat{\mathbf{n}}_z$
- Form Q's position vector from B (vector addition and rearrangement):
  $$\vec{\mathbf{r}}^{Q/B} = \vec{\mathbf{r}}^{Q/N_o} - \vec{\mathbf{r}}^{B/N_o} = 7\,\hat{\mathbf{n}}_x + {-3}\,\hat{\mathbf{n}}_y + 8\,\hat{\mathbf{n}}_z$$
- The "*vector area*" is $\vec{\Delta} \triangleq \frac{1}{2}\,\vec{\mathbf{r}}^{A/B} \times \vec{\mathbf{r}}^{Q/B} = \frac{1}{2}\,(20\,\hat{\mathbf{n}}_z) \times (7\,\hat{\mathbf{n}}_x + {-3}\,\hat{\mathbf{n}}_y + 8\,\hat{\mathbf{n}}_z) = 30\,\hat{\mathbf{n}}_x + 70\,\hat{\mathbf{n}}_y$
- The magnitude of $\vec{\Delta}$ is the area, i.e., $|\vec{\Delta}| = \sqrt{30^2 + 70^2} = 76.16$
- The unit normal $\hat{\mathbf{u}}$ in direction of $\vec{\Delta}$ is $\hat{\mathbf{u}} = \dfrac{\vec{\Delta}}{|\vec{\Delta}|} = 0.394\,\hat{\mathbf{n}}_x + 0.919\,\hat{\mathbf{n}}_y$

3.5 Proof of sine addition formula with vector cross-products

The following figure shows a generic triangle $\Delta_{OP_1Q_1}$ that has one of its angles divided into two angles, namely α and β. Two right-triangles, namely $\Delta_{OP_2Q_2}$ and $\Delta_{OQ_2R_2}$, have been constructed as a geometrical starting point for the proof that follows and as a means to provide definitions for $\cos(\alpha)$, and $\cos(\beta)$.
Note: Starting construction courtesy of Dr. Alex Perkins.



The areas of triangles $\Delta_{OP_2R_2}$, $\Delta_{OP_2Q_2}$ and $\Delta_{OQ_2R_2}$ are

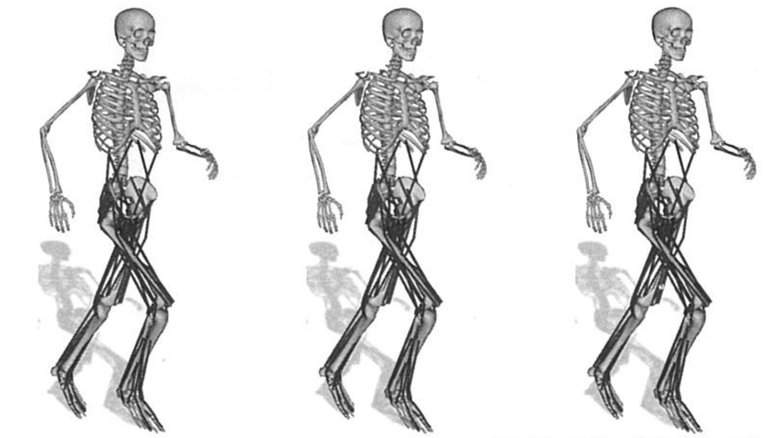
$$\text{Area}\,\Delta_{OP_2R_2} = \frac{1}{2}\,|\vec{\mathbf{OP}}_2 \times \vec{\mathbf{OR}}_2| = \frac{1}{2}\,|\vec{\mathbf{OP}}_2|\,|\vec{\mathbf{OR}}_2|\,\sin(\alpha+\beta)$$
$$\text{Area}\,\Delta_{OP_2Q_2} = \frac{1}{2}\,|\vec{\mathbf{OP}}_2 \times \vec{\mathbf{OQ}}_2| = \frac{1}{2}\,|\vec{\mathbf{OP}}_2|\,|\vec{\mathbf{OQ}}_2|\,\sin(\alpha)$$
$$\text{Area}\,\Delta_{OQ_2R_2} = \frac{1}{2}\,|\vec{\mathbf{OQ}}_2 \times \vec{\mathbf{OR}}_2| = \frac{1}{2}\,|\vec{\mathbf{OQ}}_2|\,|\vec{\mathbf{OR}}_2|\,\sin(\beta)$$

Using $\text{Area}\,\Delta_{OP_2R_2} = \text{Area}\,\Delta_{OP_2Q_2} + \text{Area}\,\Delta_{OQ_2R_2}$ and the definitions of $\cos(\alpha)$ and $\cos(\beta)$ gives

$$|\vec{\mathbf{OP}}_2|\,|\vec{\mathbf{OR}}_2|\,\sin(\alpha+\beta) = |\vec{\mathbf{OP}}_2|\,|\vec{\mathbf{OQ}}_2|\,\sin(\alpha) + |\vec{\mathbf{OQ}}_2|\,|\vec{\mathbf{OR}}_2|\,\sin(\beta)$$
$$\sin(\alpha+\beta) = \frac{|\vec{\mathbf{OQ}}_2|}{|\vec{\mathbf{OR}}_2|}\,\sin(\alpha) + \frac{|\vec{\mathbf{OQ}}_2|}{|\vec{\mathbf{OP}}_2|}\,\sin(\beta)$$
$$\sin(\alpha+\beta) = \cos(\beta)\,\sin(\alpha) + \cos(\alpha)\,\sin(\beta)$$

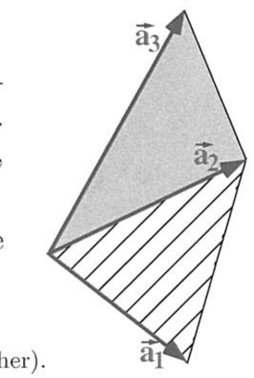
Chapter 4

Vector basis



Courtesy Dr. Sam Hamner

Why use a vector basis? (see examples in Hw 1, 2, 3)



Unit vectors are sign-posts, e.g., up, down, left, right, etc. A ***vector basis*** consisting of three orthogonal unit vectors provide a way to "give directions" in 3D space. Conventions for specifying unit vectors depend on the analyst and field of study, e.g., biomechanics, aeronautics, vehicle dynamics, statics, etc.*

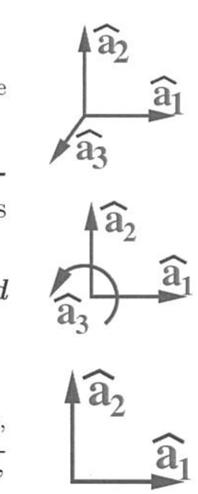
The vectors $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ shown right form a three-dimensional vector basis. Notice the basis is ***right-handed***,[a] but is not an ***orthogonal basis***[b] or ***unitary basis***.[c]

[a] The basis is ***right-handed*** (or ***dextral***) because $\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2 \cdot \vec{\mathbf{a}}_3 > 0$.
[b] An ***orthogonal basis*** has ***mutually perpendicular*** (orthogonal) basis vectors (90° to each other).
[c] A ***unitary basis*** has unit basis vectors.

One	vector basis is useful for simple directions	(e.g., point Q from point O via $\vec{\mathbf{r}}^{Q/O}$ – see Chapter 3).
Two	vector bases are useful for relative orientation	(e.g., aircraft A in Earth E via ${}^A R^E$ – see Chapter 5).
Multiple	vector bases are useful for multibody force and motion analyses.	

* For example, a vector basis for Earth's surface is ***NED*** (locally **N**orth/**E**ast/**D**own). A basis that orients Earth relative to other celestial objects is ***ECEF*** (**E**arth-**C**entered/**E**arth-**F**ixed) with a unit vector pointing from Earth's center to 0 longitude and 0 latitude, a second unit vector pointing to geometric North, and a third unit vector perpendicular to the other two.

4.1 What is a vector basis?



A ***vector basis*** is a set of linearly independent vectors that span a space (e.g., the 3D space in which we live). Each linearly independent vector is called a ***basis vector*** for the space.

It is <u>conventional</u> to use a ***right-handed basis*** and common to use a ***orthogonal unitary basis***.[a] A 3D right-handed orthogonal unitary basis has various visual representations (shown to the right). Note: When $\hat{\mathbf{a}}_3$ is absent, it is implied by the ***right-hand rule***.

Note: A set of three vectors with an intrinsic order, e.g., $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$, is called ***right-handed*** when $\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2 \cdot \vec{\mathbf{a}}_3 > 0$. Alternately, the set is ***left-handed*** when $\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2 \cdot \vec{\mathbf{a}}_3 < 0$. The orthogonal unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ are ***right-handed*** when $\hat{\mathbf{a}}_x \times \hat{\mathbf{a}}_y = \hat{\mathbf{a}}_z$.

[a] To physically demonstrate an orthogonal vector basis, hold your right hand with the thumb, forefinger, and middle finger pointing in orthogonal directions. Chapter 5 deals with ***rotation matrix*** and is summarized with two hands (each with a vector basis) and the question "how do I relate two vector bases"
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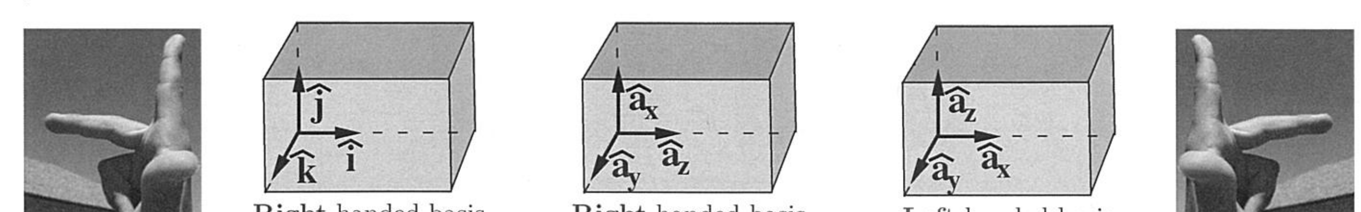

<u>Right</u>-handed basis <u>Right</u>-handed basis <u>Left</u>-handed basis